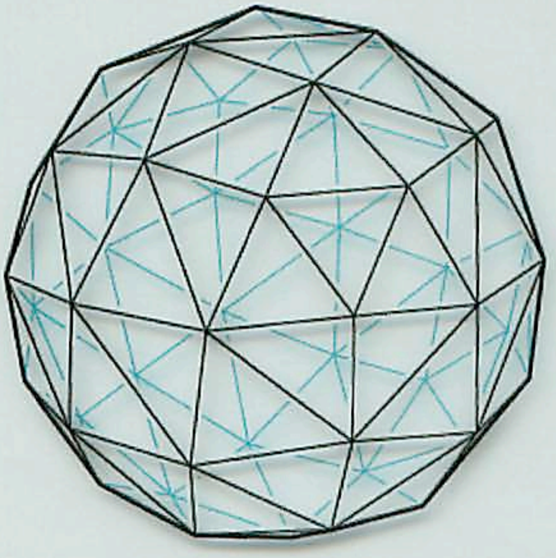


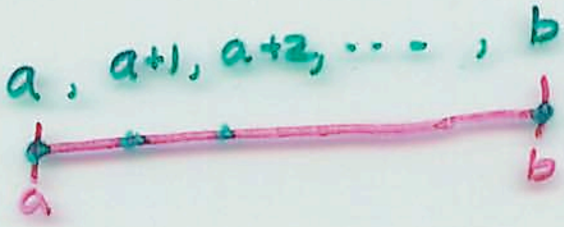
Polytope

¿ Given a polytope P with vertices at lattice points, how many lattice points are contained in P ?

In dimension one,
a polytope is....

... a line segment.

The number of lattice points in



is $b - a + 1$

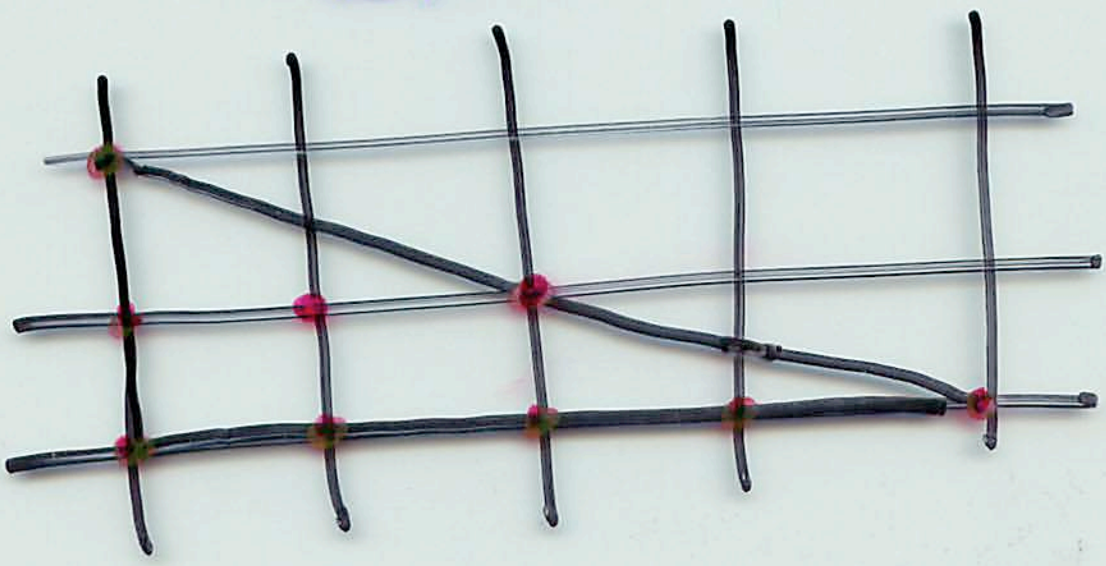
or $\text{length}(P) + 1$.

In dimension two,

$$\#(P \cap \mathbb{Z}^2) = \mathrm{Area}(\Delta) + \tfrac{1}{2}\mathrm{Perim}(\Delta) + 1$$

Pick, 1899

Ex



$$4 + \tfrac{1}{2}(4+2+2) + 1$$

$$= \boxed{9}$$

Note:

$$\#(k\Delta \cap \mathbb{Z}^2) = a_2 k^2 + a_1 k + a_0$$

$a_2 \to \mathrm{Area}(\Delta)$, $a_1 \to \tfrac{1}{2}\mathrm{Perim}(\Delta)$, $a_0 \to 1$

Ehrhart (1960's)

$\dim(\Delta) = n$

$$\#(k\Delta \cap \mathbb{Z}^n) = a_n k^n + a_{n-1} k^{n-1} + \cdots + a_0$$

$$a_i \in \mathbb{Q}$$

$$a_n = \mathrm{Vol}_n(\Delta)$$

$$a_{n-1} = \frac{1}{2} \sum_{\substack{F \subset \Delta \\ F \text{ facet}}} \mathrm{Vol}_{n-1}(F)$$

$$a_0 = \underline{1}$$

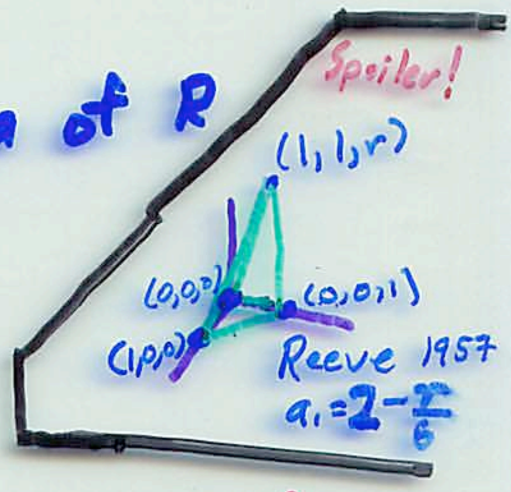
Three Dimensions

$R =$  any 3-dim polyhedron with vertices at lattice points.

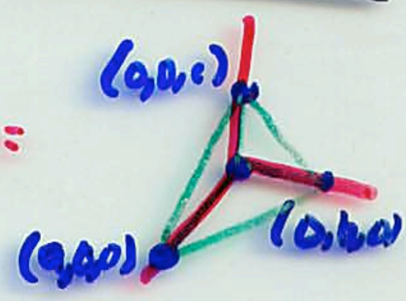
According to Ehrhart,

$$\#\text{lattice pts. in } kR = a_3 k^3 + a_2 k^2 + a_1 k + a_0$$

with
$a_3 = \text{Vol}(R)$
$a_2 = \frac{1}{2}$ Surface area of R
$a_1 =$???
$a_0 = 1$



Mordell's Tetrahedron

$R(a,b,c) =$ 

1951!
If a, b, c pairwise relatively prime, then for $R(a,b,c)$ have

$$a_1 = \frac{1}{12}\left(\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} + \frac{1}{abc}\right)$$

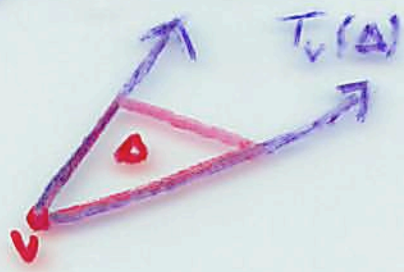
$$- s(bc, a) - s(ac, b) - s(ab, c)$$

where s is the CLASSICAL DEDEKIND SUM .....

Form of the Formula

<u>Thm</u> (McMullen, 1983)

The number of lattice points in a polytope can be expressed in terms of the volumes of its faces with coefficients depending only on the tangent cones at the faces.



More precisely, there exists a function

$$\mu : \left\{ \begin{array}{c} \text{rational } d\text{-dim} \\ \text{polyhedral cones} \\ \text{in } \mathbb{Z}^d \end{array} \right\} \to \mathbb{Q}$$

such that for any integral $\Delta \subset \mathbb{Z}^d$

$$\#(\Delta \cap \mathbb{Z}^d) = \sum_{\substack{F \text{ face} \\ \text{of } \Delta}} \mu(T_F(\Delta)) \cdot \mathrm{Vol}(F)$$

Proof non-constructive.

(Note: <u>Pick</u> does not quite have this form.)

GENERATING FUNCTIONS

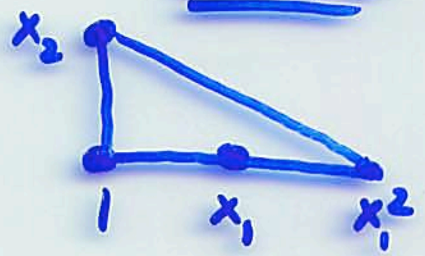
Let $\Delta \subset \mathbb{Z}^d$ be an integral polytope

Form

$$\sigma_\Delta(x_1, \ldots, x_d) = \sum_{m \in \Delta \cap \mathbb{Z}^d} x^m$$

(x^m shorthand for $x_1^{m_1} x_2^{m_2} \ldots x_d^{m_d}$.)

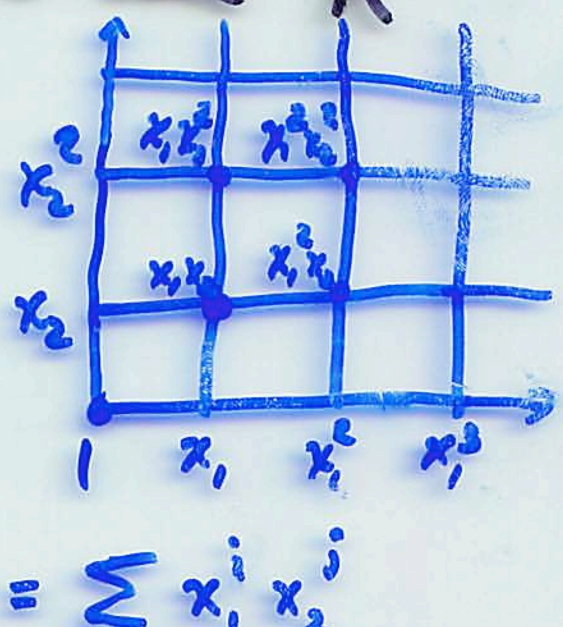
Example



$$\sigma_\Delta = 1 + x_1 + x_1^2 + x_2$$

Can do the same for a cone K

$$\sigma_K(x_1, \ldots x_d) = \sum_{m \in K \cap \mathbb{Z}^d} x^m$$



$$\sigma_K = \sum_{i,j} x_1^i x_2^j$$

$$= \frac{1}{1-x_1} \; \frac{1}{1-x_2}$$

An Important Curiosity

Sum over a whole line L

$$\sigma_L = \frac{x^{-1}}{1-x^{-1}} + \frac{1}{1-x}$$

$$= \underline{\underline{0}}$$

Similarly, summing over the lattice points in any <u>half-space</u> gives 0.

$$\sigma_H = 0.$$

Theorem (Brion, 1988)

Δ integral convex polytope

$$\sigma_\Delta = \sum_{\substack{v \text{ vertex} \\ \text{of } \Delta}} \sigma_{K_v} .$$

→ original proof used K-theory of toric varieties.

Example

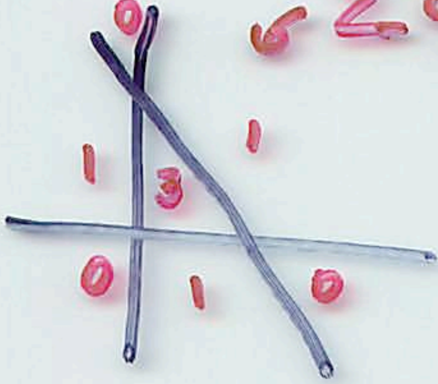
has two vertex cones

$$\frac{1}{1-x} + \frac{x^k}{1-x^{-1}}$$

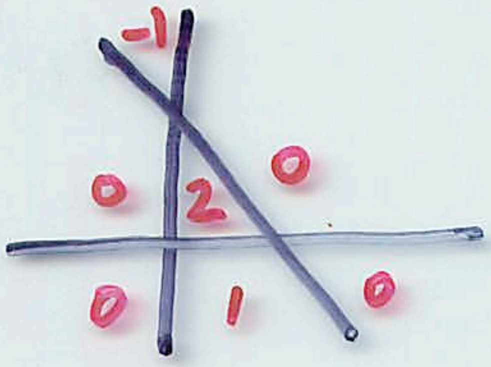
$$= \frac{1-x^{k+1}}{1-x} = 1 + x + \cdots + x^k$$

"Proof" of Brion

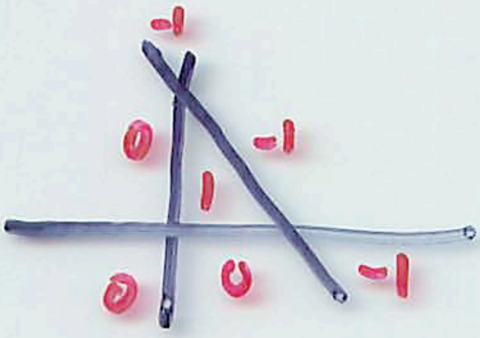
$\curvearrowleft \sum \sigma_{K_v}$



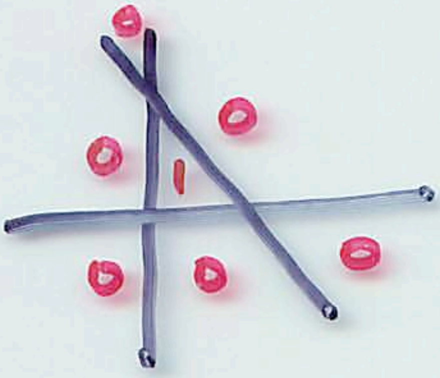
subtract



subtract



add



$\rightarrow \sigma_\Delta$

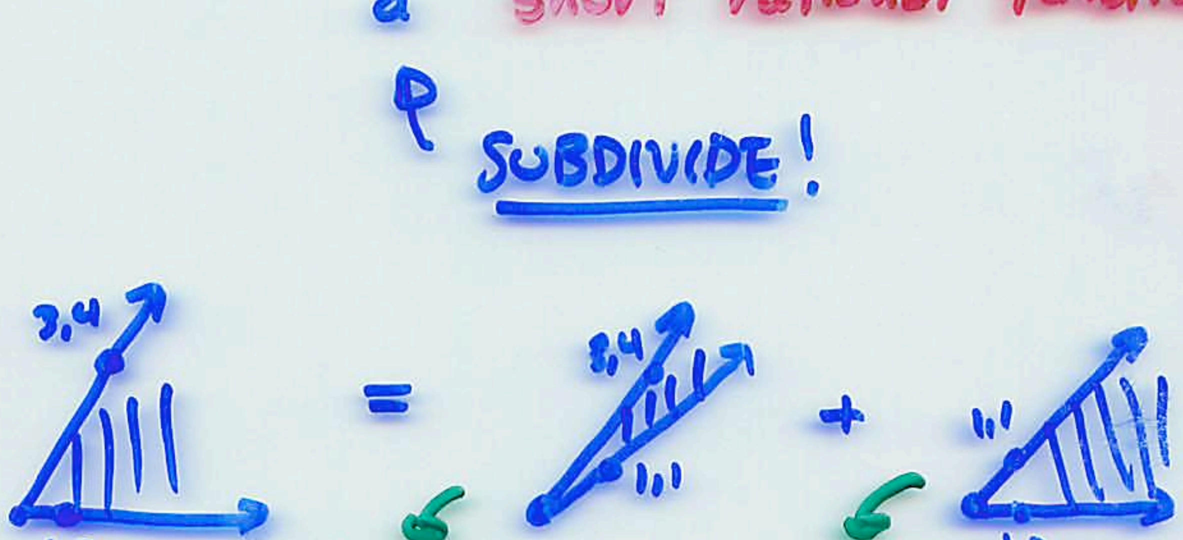
Theorem (Barvinok, 1993)

Brion's Theorem can be used to give a polynomial-time algorithm to compute the number of lattice points in a polytope of fixed dimension.

Idea: → Brion reduces problem to computation of σ_{K_v} (K_v = tangent cones at vertices)

→ σ_{K_v} has an expression as a short rational function.

& SUBDIVIDE!



$$\frac{1}{1-x_1x_2}\cdot\frac{1}{1-x_1^3x_2^4} \qquad \frac{1}{1-x_1}\cdot\frac{1}{1-x_1x_2}$$

TWO KINDS OF ADDITIVITY

M-ADDITIVITY (NAIVE)

$\tau = \tau_1 \cup \tau_2$

$$\sum_{\tau} x^m = \sum_{\tau_1} x^m + \sum_{\tau_2} x^m - \sum_{\rho} x^m$$

N-ADDITIVITY

DUAL CONE:

$N = \mathrm{Hom}(N, \mathbb{Z})$

SINCE HALF-PLANE SUMS VANISH,

$$\sum_{\check{\sigma}} x^m = \sum_{\check{\sigma}_1} x^m + \sum_{\check{\sigma}_2} x^m$$

ON THE NOSE (NO INCLUSION-EXCLUSION!)

Toric Varieties, briefly

K = cone w/vertex at 0

Observation: $K \cap \mathbb{Z}^d$ is a semigroup.

The semigroup algebra $\mathbb{C}[K \cap \mathbb{Z}^d]$
is a finitely generated algebra

$\rightsquigarrow$ affine algebraic variety
$X_K = \mathrm{Spec}(\mathbb{C}[K \cap \mathbb{Z}^d])$

Ex

$\rightsquigarrow \mathbb{C}[K \cap \mathbb{Z}^2] = \mathbb{C}[X,Y] \rightsquigarrow X_K = \mathbb{C}^2$

Ex

$\rightsquigarrow \mathbb{C}[K \cap \mathbb{Z}^2] = \dfrac{\mathbb{C}[X,Y,Z]}{Y^2 - XZ} \rightsquigarrow X_K =$ 

$X_K = \{(x,y,z) \in \mathbb{C}^3 \mid y^2 = xz\}$

For any cone K we have an affine algebraic variety X_K.

If Δ is an integral polytope, the tangent cones $\{K_v \mid v \text{ vertex of } \Delta\}$ yield affine varieties $\{X_{K_v} \mid v \text{ vertex of } \Delta\}$.

These glue together to form a projective variety $X_\Delta = \bigcup X_{K_v}$

X_Δ is called the toric variety associated to the polytope Δ.

Counting Lattice Points using Toric Varieties

Turns out:

$$\#(P \cap \mathbb{Z}^d) \longleftrightarrow \mathrm{Td}\, X_P \in H_*(X_P, \mathbb{Q})$$

"Todd class"

Reason: Riemann-Roch

Details: Can introduce a line bundle $\mathcal{E}_P$ on X_P such that

$$\left\{ \begin{matrix} \text{basis of sections} \\ \text{of } \mathcal{E}_P \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{lattice points} \\ \text{inside } P \end{matrix} \right\}$$

$$\#(P \cap \mathbb{Z}^d) = \dim H^0(X_P, \mathcal{E}_P) \underset{\substack{\uparrow \\ \text{higher cohomology} \\ \text{vanishes}}}{=} \sum (-1)^i \dim H^i(X_P, \mathcal{E}_P) \underset{\substack{\uparrow \\ \text{R.R.}}}{=} (\mathrm{ch}\, \mathcal{E}_P, \mathrm{Td}\, X_P)$$

Consequence:

$$\mathrm{Td}\, X_P = \sum r_F [V(F)] \Longrightarrow \#(P \cap \mathbb{Z}^d) = \sum r_F \,\mathrm{Vol}(F)$$

MAIN DEFINITION

Let $\sigma = \langle \rho_1, \dots \rho_n \rangle$ be an n-dim. (simplicial) cone in N.

DEF

$$s_\sigma(x_1, \dots x_n) = \sum_{m \in \check{\sigma}} e^{-(\langle m, \rho_1 \rangle x_1 + \cdots \langle m, \rho_n \rangle x_n)}$$

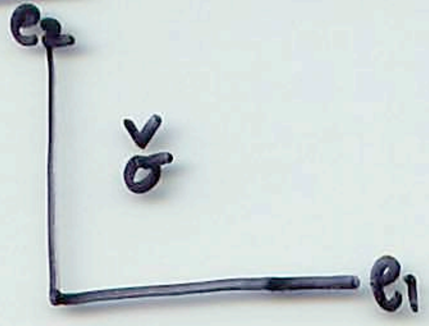
(a minor variant of $\sum_{\check{\sigma}} e^{-m}$)

It's cousin

$$t_\sigma(x_1, \dots x_n) = x_1 x_2 \cdots x_n \, s_\sigma(x_1, \dots x_n)$$

is actually a power series in $x_1, \dots x_n$

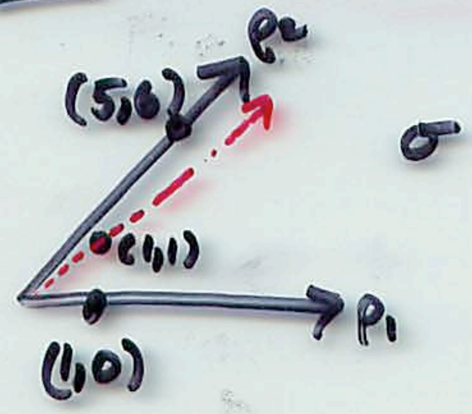
EX 1 A NONSINGULAR CONE

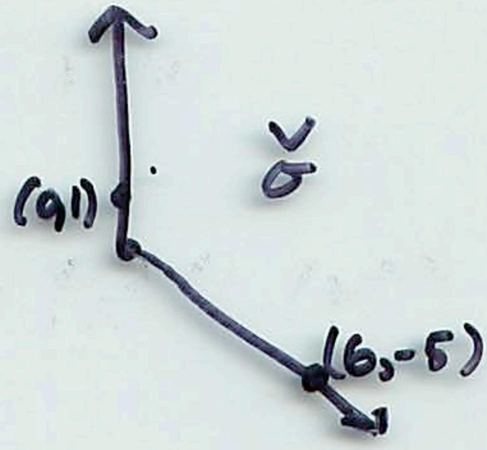


$$s_\sigma = \frac{1}{1-e^{-x_1}} \frac{1}{1-e^{-x_2}}$$

$$t_\sigma(x_1, x_2) = \frac{x_1}{1-e^{-x_1}} \frac{x_2}{1-e^{-x_2}}$$
$$= 1 + \frac{x_1}{2} + \frac{x_2}{2} + \frac{1}{4} x_1 x_2 + \frac{1}{12}(x_1^2 + x_2^2) + \cdots$$

Ex 2





$$S_\sigma(x_1, x_2) = \frac{1}{1-e^{y-x}}\,\frac{1}{1-e^{-6y}} + \frac{1}{1-e^{-6x}}\,\frac{1}{1-e^{x-y}}$$

$$= \frac{1}{1-e^{-6x}}\,\frac{1}{1-e^{-6y}}\left[1 + e^{-(5x+y)} + e^{-(4x+2y)} + e^{-(3x+3y)} + e^{-(2x+4y)} + e^{-(x+5y)}\right]$$

$$t_\sigma(x,y) =$$

$$1 + \frac{x}{2} + \frac{y}{2} + \frac{1}{12}(x^2+y^2) + \left(\frac{-49}{6}\right)xy + \cdots$$

Dedekind sum.

PROPERTIES OF s_σ

- For nonsingular cones σ

$$s_\sigma(x_1, \dots x_n) = \frac{1}{1-e^{-x_1}} \cdots \frac{1}{1-e^{-x_n}}$$

- s_σ is additive with respect to subdivisions

  ($\underline{\underline{N}}$-ADDITIVE)

  $\sigma = \cup \sigma_i$ mod smaller dim cones

  $\Rightarrow \quad s_\sigma = \sum s_{\sigma_i}$

  with appropriate coordinate changes.

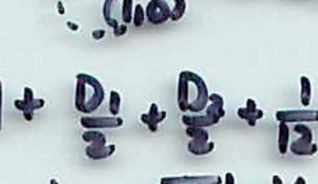
- [P, '96]

The t_σ determine a local formula for the Todd class of any simplicial toric variety

$$\mathrm{Td}\, X_\Delta = \sum_{\substack{F_1, \dots F_k \\ \text{facets}}} t_{F_1, \dots F_k} [V(F_1)]^{a_1} \cdots [V(F_k)]^{a_k}$$

↑ polynomial in torus invariant divisors

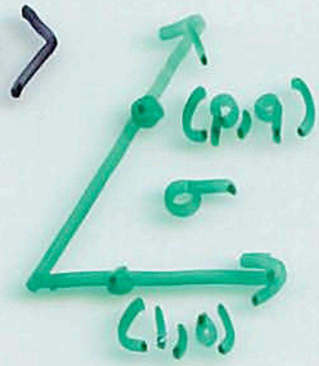
Ex If Δ has this cone, so the normal fan is



then $1 + \frac{D_1}{2} + \frac{D_2}{2} + \frac{1}{12}(D_1^2 + D_2^2) - \frac{49}{6} D_1 D_2 + \cdots$ appear in $\mathrm{Td}\, X_\Delta$.

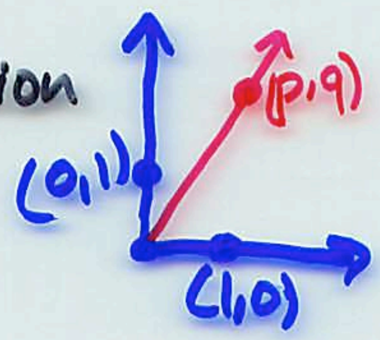
....and Todd class formulas:

Thm If σ is a two-dimensional cone isomorphic to $\langle (1,0), (p,q) \rangle$ in $\mathbb{Z}^2$, then



$$t_\sigma(x,y) = 1 + \frac{1}{2}(x+y) + \frac{1}{12}(x^2+y^2)$$
$$+ \; q\left(\underline{s(p,q)} + \frac{1}{4}\right) xy + \cdots \text{higher degree terms}$$

REMARKS:

- The subdivision  gives Dedekind reciprocity

- more general subdivisions $\longrightarrow$ new reciprocity laws

- higher degree terms $\longrightarrow$ "higher dimensional Dedekind sums"

$\circ \quad \circ \quad \circ$

Lattice Point Consequences

1) For $d=3$, have a_1 expressed in terms of classical Dedekind sums.
(P.'93) For any d, $a_{d-2}(\Delta) \rightsquigarrow$ Dedekind sums.

2) For higher d, all Ehrhart coefficients expressed in terms of generalized Dedekind sums.
Brion-Vergne '97
Diaz-Robins '96
P. '95 → reciprocity relations

3) Poly-time computability of formulas of Brion-Vergne, Diaz-Robins, Cappell-Shaneson, and of Zagier's higher-dim. Dedekind sums.
(Barvinok-P. '98)

UNEXPECTED:

4) For $d=2$, have $a_0=1$ expressed in terms of Dedekind sums

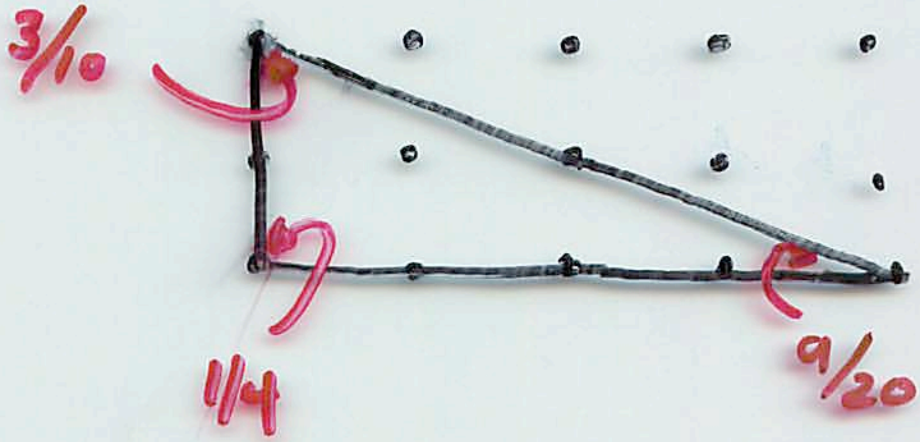
$\Downarrow$

new reciprocity relations for Dedekind sums !!

5) Constructive McMullen-type formulas for any lattice with inner product.

(P.- Thomas '03)

(Turns out ....



)

→ Extended to Euler-Maclaurin formulas

Berline-Vergne '06

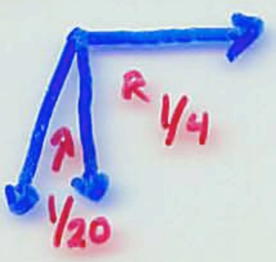
μ () = ?

Recipe:

- First dualise.

- Subdivide into unimodular cones.

- Use formula $\frac{1}{4} - \frac{1}{12}\frac{v_1 \cdot v_2}{v_1 \cdot v_1} - \frac{1}{12}\frac{v_1 \cdot v_2}{v_2 \cdot v_2}$.



- Add.

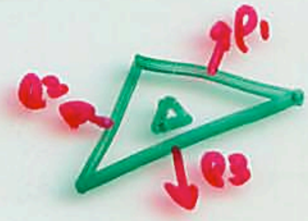
$$\frac{1}{20} + \frac{1}{4} = \frac{3}{10}$$

II BRION'S TODD OPERATOR

Again suppose $\Delta \subset M$ is a lattice polytope:

$$\Delta = \{ m \in M \mid \langle \rho_i, m \rangle \geq b_i \}_{i=1\ldots k}$$

primitive normals to facets of Δ.



Now DEFORM Δ as follows:

$$\Delta(h) = \{ m \in M \mid \langle \rho_i, m \rangle \geq b_i - h_i \}_{i=1\ldots k}$$

THM (BRION–VERGNE) Δ simple.
For any polynomial function ϕ on M,

$$\sum_{m \in \Delta \cap M} \phi(m) = t_\Delta\left(\partial/\partial h_1, \ldots \partial/\partial h_k\right) \int_{\Delta(h)} \phi \,\Bigg|_{h=0}.$$

power series whose restriction to any face of Δ corresponds to t_σ.

In particular, $\phi \equiv 1$ counts lattice pts:

$$\#(\Delta \cap M) = t_\Delta\left(\partial/\partial h_1, \dots, \partial/\partial h_d\right) \operatorname{Vol}(\Delta(h))\Big|_{h=0}.$$

The case when Δ is non-singular is due to Khovanskii.

$$f(t) \sim \frac{c}{t} + c_0 + c_1 t + c_2 t^2 + \cdots$$

Then: (1) φ extends to an analytic function on $\mathbb{C}$ (with a simple pole at $s=1$)

(2) $\varphi(-n) = (-1)^n n! \, c_n$

Key lemma of Zagier:

LEMMA Let

$$\varphi(s) = \frac{a_1}{\lambda_1^s} + \frac{a_2}{\lambda_2^s} + \cdots\cdots$$

$\lambda_i \in \mathbb{R}^+ \quad \lambda_n \to \infty$ as $n \to \infty$

Form the corresponding exponential series

values of zeta functions at negative integers. (Joint work with S. Garoufalidis)

$$f(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + \cdots\cdots$$

$K = \mathbb{Q}[\sqrt{d}]$

Supposing that as $t \to 0^+$

$\mathcal{O}_K$ = ring of integers

V = group of totally positive units

Relation with toric varieties:

$\mathcal{O}_K/V$ has as a fundamental domain a two-dimensional cone!

So to compute zeta function values, need a way to sum a function over a cone.

Fortunately we have the following version of Brion's formula:

THM If $\sigma = \langle p_1, \ldots, p_n \rangle$ is an n-dim simplicial cone in $N = \mathrm{Hom}(M, \mathbb{Z})$ and $\phi : M \to \mathbb{C}$ is a suitably rapidly decreasing analytic function, then

$$\sum_{m \in \check{\sigma} \cap M} \phi(m) = t_\sigma\left(\frac{\partial}{\partial h_1}, \ldots, \frac{\partial}{\partial h_n}\right) \int_{\check{\sigma}(h)} \phi \,\Bigg|_{h=0}$$

Here $\check{\sigma}(h) = \{ m \in M \mid \langle p_i, m \rangle \geq -h_i \}$.

Relation with toric varieties:

$\mathcal{O}_K/V$ has as a fundamental domain a two-dimensional cone!

So to compute zeta function values, need a way to sum a function over a cone.

Fortunately we have the following version of Brion's formula:

<u>THM</u> If $\sigma = \langle \rho_1, \dots, \rho_n \rangle$ is an n-dim simplicial cone in $N = \mathrm{Hom}(M, \mathbb{Z})$ and $\phi : M \to \mathbb{C}$ is a suitably rapidly decreasing analytic function, then

$$\sum_{m \in \check{\sigma} \cap M} \phi(m) = t_\sigma\left(\frac{\partial}{\partial h_1}, \dots, \frac{\partial}{\partial h_n}\right) \int_{\check{\sigma}(h)} \phi \, \Bigg|_{h=0}$$

Here $\check{\sigma}(h) = \{ m \in M \mid \langle \rho_i, m \rangle \geq -h_i \}$.

Key lemma of Zagier:

<u>LEMMA</u> Let

$$\varphi(s) = \frac{a_1}{\lambda_1^s} + \frac{a_2}{\lambda_2^s} + \cdots\cdots$$

$\lambda_i \in \mathbb{R}^+$ $\lambda_n \to \infty$ as $n \to \infty$

Form the corresponding exponential series

$$f(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + \cdots\cdots$$

Supposing that as $t \to 0^+$

$$f(t) \sim \frac{c}{t} + c_0 + c_1 t + c_2 t^2 + \cdots$$

Then: (1) φ extends to an analytic function on $\mathbb{C}$ (with a simple pole at $s=1$)

(2) $\varphi(-n) = (-1)^n \, n! \, c_n$

We thus recover and reexamine some classical formulas with the modern lattice point techniques.

COEFFS OF t_σ ARE CHARACTERIZED BY RECIPROCITY LAW $\Rightarrow$ SHINTANI-TYPE FORMULAS FOR $\zeta(-n)$

COEFFS OF t_σ GIVEN BY CYCLOTOMIC SUMS $\Rightarrow$ ZAGIER-TYPE FORMULAS FOR $\zeta(-n)$

For REAL QUADRATIC number fields,

cones and quadratic forms are parametrized

by a sequence

$$[b_0, b_1, \dots b_{r-1}]$$

of integers with $b_i > 1$, and not all $b_i = 2$.

Set $\quad b_k = b_{k \bmod r}$ for any $k \in \mathbb{Z}$.

Define $\quad w_k = b_k - \cfrac{1}{b_{k+1} - \cfrac{1}{b_{k+2} - \dots}}$

w_0 satisfies quadratic equation $Aw^2 + Bw + C = 0$
w/ discriminant
$D = B^2 - 4AC$.

Set $\quad A_0 = 1$ and

$$A_{k-1} = A_k w_k$$

It follows that

$$A_{k-1} + A_{k+1} = b_k A_k$$

Let $M = \mathbb{Z} \oplus \mathbb{Z} w_0 \subset \mathbb{Q}(\sqrt{D})$ 2-dim lattice

and $Q : M_{\mathbb{R}} \to \mathbb{R}$

by $\quad Q(xw_0 + y) = Cx^2 - Bxy + Ay^2$

Example 1.2. We will express $\zeta_{Q,\tau}(-1)$ and $\zeta_{Q,\tau}(-2)$ using Theorem 1. For $i = 0,\ldots,r-1$, we define L_i, M_i, N_i to be the coefficients of the quadratic form Q on the i^{th} nonsingular cone $\langle A_{i-1}, A_i\rangle$. Explicitly,

$$Q(xA_{i-1}+yA_i) = L_i x^2 + M_i xy + N_i y^2.$$

We define $\tilde{L}_i, \tilde{M}_i, \tilde{N}_i$ similarly, as the coefficients of Q on the cone $\langle A_{i-1}, A_{i+1}\rangle$, generated by rays two apart:

$$Q(xA_{i-1}+yA_{i+1}) = \tilde{L}_i x^2 + \tilde{M}_i xy + \tilde{N}_i y^2.$$

Note that for sequences b of fixed length r, $L_i, M_i, N_i, \tilde{L}_i, \tilde{M}_i, \tilde{N}_i$ are polynomials in b_i with integer coefficients, as follows from Lemma 3.2. Theorem 1 then gives us:

$$\zeta_{Q,\tau}(-1) = \frac{1}{720}\sum_{i=0}^{r-1}(5M_i + b_i(-2\tilde{L}_i + \tilde{M}_i - 2\tilde{N}_i)).$$

We may compare this with a formula of Zagier [Za4, p.149], which involves only the L_i, M_i, N_i and not the $\tilde{L}_i, \tilde{M}_i, \tilde{N}_i$, though it does involve higher powers of the b_i:

$$\zeta_{Q,\tau}(-1) = \frac{1}{720}\sum_{i=0}^{r-1}(-2N_i b_i^3 + 3M_i b_i^2 - 6L_i b_i + 5M_i).$$

The patient reader may use Lemma 3.2 to show that the above two expressions for $\zeta_{Q,\tau}$ are the same polynomial in the b_i with rational coefficients.
As for $\zeta_{Q,\tau}(-2)$, Theorem 1 yields the following expression:

$$\zeta_{Q,\tau}(-2) = \frac{1}{15120}\sum_{i=0}^{r-1}(-21M_i(L_i+N_i)+2b_i(6\tilde{L}_i^2-3\tilde{L}_i\tilde{M}_i+2\tilde{L}_i\tilde{N}_i+\tilde{M}_i^2-3\tilde{M}_i\tilde{N}_i+6\tilde{N}_i^2)).$$

toric geometry which are necessary for the proof and which lead to a conceptual understanding of the present formula.

Let λ_m be defined by the power series:

$$\frac{h}{1-e^{-h}} = \sum_{m=0}^{\infty} \lambda_m h^m \tag{4}$$

thus we have: $\lambda_m = (-1)^m B_m/m!$ where B_m is the m^{th} Bernoulli number. (See also Definition 1.6 below.) Note that if $m > 1$ is odd, then $\lambda_m = 0$. For $n \geq 0$, define homogeneous polynomials $P_n(X,Y), R_n(X,Y)$ of degree $2n$ by:

$$\begin{aligned} P_n(X,Y) &= \sum_{i+j=2n, i,j\geq 0} (-1)^{i+1}\lambda_{i+1}\lambda_{j+1}X^iY^j, \\ R_n(X,Y) &= \frac{X^{2n+1}+Y^{2n+1}}{X+Y} = X^{2n} - X^{2n-1}Y + \cdots + Y^{2n}. \end{aligned}$$

We then have:

Theorem 1. *For a sequence b, as above, with associated (M,Q,τ), the values $\zeta_{Q,\tau}(-n)$ for $n \geq 0$ are given explicitly as follows:*

$$\zeta_{Q,\tau}(-n) = P_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \sum_{i=0}^{r-1} (Q(xA_{i-1}+yA_i)^n) \tag{5}$$

$$+\lambda_{2n+2}R_n\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \sum_{i=0}^{r-1} b_i(Q(xA_{i-1}+yA_{i+1})^n). \tag{6}$$

If the length r of the sequence b is fixed, the above expresses $\zeta_{Q,\tau}(-n)$ as a polynomial in the b_i with rational coefficients, symmetric under cyclic permutation of the b_i.

In particular, we obtain the formula due to Meyer, see also [Za1, Equation 3.3]:

$$\zeta_{Q,\tau}(0) = \frac{1}{12}\sum_{i=0}^{r-1}(b_i - 3). \tag{7}$$