Numerical computation of Chow-Heegner points associated to pairs of elliptic curves[*][†]

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Abstract

In this paper, we consider a special case of Chow-Heegner points that has a simple concrete description due to Shouwu Zhang. Given a pair E, F of nonisogenous elliptic curves, and surjective morphisms $\varphi_E : X_0(N) \to E$ and $\varphi_F : X_0(N) \to F$ of curves over $\mathbb{Q}$, we associate a rational point $P \in E(\mathbb{Q})$. We describe a numerical approach to computing P, state some motivating results of Zhang et al. about the height of P, and present a table of data.

1 Introduction: Zhang's Construction

Consider a pair E, F of nonisogenous elliptic curves over $\mathbb{Q}$ and fix surjective morphisms from $X_0(N)$ to each curve. We do *not* assume that N is the conductor of either E or F, though N is necessarily a multiple of the conductor.

$$\begin{tikzcd} & X_0(N) \arrow[dl, "\varphi_E"'] \arrow[dr, "\varphi_F"] & \\ E & & F \end{tikzcd}$$

Let $(\varphi_E)_* : \mathrm{Div}(X_0(N)) \to \mathrm{Div}(E)$ and $\varphi_F^* : \mathrm{Div}(F) \to \mathrm{Div}(X_0(N))$ be the pushforward and pullback maps on divisors on algebraic curves. Let $Q \in F(\mathbb{C})$ be any point, and let

$$P_{\varphi_E,\varphi_F,Q} = \sum (\varphi_E)_* \varphi_F^*(Q) \in E(\mathbb{C}),$$

where $\sum$ means the sum of the points in the divisor using the group law on E, i.e., given a divisor $D = \sum n_i P_i \in \mathrm{Div}(E)$, we have $(\sum D) - \infty \sim D - \deg(D)\infty$, which uniquely determines $\sum D$.

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[†]A modified version of this paper will be published as an appendix to [DDLR11].

Proposition 1.1. *The point $P_{\varphi_E,\varphi_F,Q}$ does not depend on the choice of Q.*

Proof. The composition $(\varphi_E)_* \circ \varphi_F^*$ induces a homomorphism of elliptic curves

$$\psi : \mathrm{Pic}^0(F) = \mathrm{Jac}(F) \to \mathrm{Jac}(E) = \mathrm{Pic}^0(E).$$

Our hypothesis that E and F are nonisogenous implies that $\psi = 0$. We denote by $[D]$ the linear equivalence class of a divisor in the Picard group. If $Q' \in F(\mathbb{C})$ is another point, then under the above composition of maps,

$$[Q - Q'] \mapsto [(\varphi_E)_* \varphi_F^*(Q) - (\varphi_E)_* \varphi_F^*(Q')] = [P_Q - P_{Q'}].$$

Thus the divisor $P_Q - P_{Q'}$ is linearly equivalent to 0. But F has genus 1, so there is no rational function on F of degree 1, hence $P_Q = P_{Q'}$, as claimed. □

We let $P_{\varphi_E,\varphi_F} = P_{\varphi_E,\varphi_F,Q} \in E(\mathbb{C})$, for any choice of Q.

Corollary 1.2. *We have $P_{\varphi_E,\varphi_F} \in E(\mathbb{Q})$.*

Proof. Taking $Q = \mathcal{O} \in F(\mathbb{Q})$, we see that the divisor $(\varphi_E)_* \circ \varphi_F^*(Q)$ is rational, so its sum is also rational. □

In the rest of this paper, we write $P_{E,F} = P_{\varphi_E,\varphi_F}$ when E and F are both optimal curves of the same conductor N, and φ_E and φ_F are as in Section 5.

1.1 Outline

In Section 2 we discuss an example in which E and F both have conductor 37. Section 3 is about a formula of Yuan-Zhang-Zhang for the height of $P_{E,F}$ in terms of the derivative of an L-function, in some cases. In Section 4, we discuss the connection between this paper and the paper [DDLR11] about computing Chow-Heegner points using iterated integrals. The heart of the paper is Section 5, which describes our numerical approach to approximating $P_{E,F}$. Finally, Section 5.2 presents a table of points $P_{E,F}$.

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2 Example: $N = 37$

The smallest conductor curve of rank 1 is the curve E with Cremona label 37a (see [Cre]). The paper [MSD74] discusses the modular curve $X_0(37)$ in detail.

It gives the affine equation $y^2 = -x^6 - 9x^4 - 11x^2 + 37$ for $X_0(37)$, and describes how $X_0(37)$ is equipped with three independent involutions w, T and S. The quotient of $X_0(37)$ by w is E, the quotient by T is an elliptic curve F with $F(\mathbb{Q}) \approx \mathbb{Z}/3\mathbb{Z}$ and Cremona label 37b, and the quotient by S is the projective line $\mathbb{P}^1$.

$$\begin{tikzcd}
 & X_0(37) \arrow[dl, "\varphi_E"'] \arrow[d, "\varphi_F"] \arrow[dr] & \\
E = X/w & F = X/T & \mathbb{P}^1 = X/S
\end{tikzcd}$$

The maps φ_E and φ_F have degree 2, by virtue of being induced by an involution. As explained in [MSD74], the fiber over $Q = 0 \in F(\mathbb{Q})$ contains 2 points:

1. the cusp $[\infty] \in X_0(37)(\mathbb{Q})$, and
2. the noncuspidal affine rational point $(-1, -4) = T(\infty) \in X_0(37)(\mathbb{Q})$.

We have $\varphi_E([\infty]) = 0 \in E(\mathbb{Q})$, and [MSD74, Prop. 3, pg. 30] implies that

$$\varphi_F((-1, -4)) = (6, 14) = -6(0, -1),$$

where $(0, -1)$ generates $E(\mathbb{Q})$. We conclude that

$$P_{E,F} = (6, 14) \qquad \text{and} \qquad [E(\mathbb{Q}) : \mathbb{Z}P_{E,F}] = 6.$$

On [MSD74, pg. 31], they remark: "It would be of the utmost interest to link this index to something else in the theory."

This remark motivates our desire to compute more examples. Unfortunately, it is very difficult to generalize the above approach directly, since it involves computations with $X_0(37)$ and its quotients that rely on explicit defining equations. Just as there are multiple approaches to computing Heegner points, there are several approaches to computing $P_{E,F}$:

- a Gross-Zagier style formula for the height of $P_{E,F}$ (see Section 3),
- explicit evaluation of iterated integrals (see Section 4), and
- numerical approximation of the fiber in the upper half plane over a point on F using a polynomial approximation to φ_F (see Section 5).

This paper is mainly about the last approach listed above.

3 The Formula of Yuan-Zhang-Zhang

Consider a special case of the triple product L-function of [GK92]

$$L(E, F, F, s) = L(E, s) \cdot L(E, \mathrm{Sym}^2(F), s), \tag{1}$$

where E and F are elliptic curves of the same conductor N, and all L-functions are normalized so that $1/2$ is the center of the critical strip. The following theorem is proved in [YZZ11]:

Theorem 3.1 (Yuan-Zhang-Zhang)**.** *Assume that the local root number of $L(E,F,F,s)$ at every prime of bad reduction is $+1$ and that the root number at infinity is -1. Then $\hat{h}(P_{E,F}) = (*) \cdot L'(E,F,F,\frac{1}{2})$, where $(*)$ is nonzero.*

Remark 3.2. The above formula resembles the Gross-Zagier formula

$$\hat{h}(P_K) = (*) \cdot (L(E/\mathbb{Q},s) \cdot L(E^K/\mathbb{Q},s))'|_{s=\frac{1}{2}},$$

where K is a quadratic imaginary field satisfying certain hypotheses.

If one could evaluate $L'(E,F,F,\frac{1}{2})$, e.g., by applying the algorithm of [Dok04], along with the factor $(*)$ in the theorem, this would yield an algorithm to compute $\pm P_{E,F} \pmod{E(\mathbb{Q})_{\mathrm{tor}}}$ when the root number hypothesis is satisfied. Unfortunately, it appears that nobody has numerically evaluated the formula of Theorem 3.1 in any interesting cases.

When E and F have the same squarefree conductor N, [GK92, §1] implies that the local root number of $L(E,F,F,s)$ at p is the same as the local root number of E at p; computing the local root number when the level is not square free is more complicated.

Proposition 3.3. *Assume that E and F have the same squarefree conductor N, that the local root numbers of E at primes $p \mid N$ are all $+1$ (equivalently, that we have $a_p(E) = -1$) and that $r_{\mathrm{an}}(E/\mathbb{Q}) = 1$. Then $L(E, \mathrm{Sym}^2 F, \frac{1}{2}) \neq 0$ if and only if $\hat{h}(P_{E,F}) \neq 0$.*

Proof. By hypothesis, we have $L(E,\frac{1}{2}) = 0$ and $L'(E,\frac{1}{2}) \neq 0$. Theorem 3.1 and the factorization (1) imply that

$$\hat{h}(P_{E,F}) = (*) \cdot L'(E,\frac{1}{2}) \cdot L(E, \mathrm{Sym}^2 F, \frac{1}{2}),$$

from which the result follows. □

Section 5.2 contains numerous examples in which E has rank 1, F has rank 0, and yet $P_{E,F}$ is a torsion point. The first example is when E is 91b and F is 91a. Then $P_{E,F} = (1,0)$ is a torsion point (of order 3). In this case, we cannot apply Proposition 3.3 since $\varepsilon_7 = \varepsilon_{13} = -1$ for E. Another example is when E is 99a and F is 99c, where we have $P_{E,F} = 0$, and $\varepsilon_3 = \varepsilon_{11} = +1$, but Proposition 3.3 does not apply since the level is not square free. Fortunately, we found an example with squarefree level $158 = 2 \cdot 79$: here E is 158b, F is 158d, we have $P_{E,F} = 0$ and $\varepsilon_2 = \varepsilon_{79} = +1$, so Proposition 3.3 implies that $L(E, \mathrm{Sym}^2 F, \frac{1}{2}) = 0$.

4 Iterated Complex Path Integrals

The paper [DDLR11] contains a general approach using iterated path integrals to compute certain Chow-Heegner points, of which $P_{E,F}$ is a specific instance. Comparing our data (Section 5.2) with theirs, we find that if E and F are optimal elliptic curves over $\mathbb{Q}$ of the same conductor $N \leq 100$, if $e, f \in S_2(\Gamma_0(N))$

are the corresponding newforms, and if $P_{f,e,1} \in E(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}$ the associated Chow-Heegner point in the sense of [DDLR11], then $2P_{E,F} = P_{f,e,1}$. This is (presumably) a consequence of [DRS11].

5 A Numerical Approach to Computing $P_{E,F}$

The numerical approach to computing P that we describe in this section uses relatively little abstract theory. It is inspired by work of Delaunay (see [Del02]) on computing the fiber of the map $X_0(389) \to E$ over rational points on the rank 2 curve E of conductor 389. We make no guarantee about how many digits of our approximation to $P_{E,F}$ are correct, instead viewing this as an algorithm to produce something that is useful for experimental mathematics only.

Let $\mathfrak{h}$ be the upper half plane, and let $Y_0(N) = \Gamma_0(N)\backslash\mathfrak{h} \subset X_0(N)$ be the affine modular curve. Let E and F be nonisogenous optimal elliptic curve quotients of $X_0(N)$, with modular parametrization maps φ_E and φ_F, and assume both Manin constants are 1. Let Λ_E and Λ_F be the period lattices of E and F, so $E \cong \mathbb{C}/\Lambda_E$ and $F \cong \mathbb{C}/\Lambda_F$. Viewed as a map $[\tau] \mapsto \mathbb{C}/\Lambda_E$, we have, using square brackets to denote equivalence classes, that

$$\varphi_E([\tau]) = \left[\sum_{n=1}^{\infty} \frac{a_n}{n} e^{2\pi i n\tau}\right],$$

and similarly for φ_F, where $a_n = a_n(E)$ are the L-series coefficients of E (see [Cre97, §2.10], which uses the oppositive sign convention). For any positive integer B, define the polynomial

$$\varphi_{E,B} = \sum_{n=1}^{B} \frac{a_n}{n} T^n \in \mathbb{Q}[T],$$

and similarly for $\varphi_{F,B}$.

To approximate $P_{E,F}$, we proceed as follows. First we make some choices, and after making these choices we run the algorithm, which will either find a "probable" numerical approximation to $P_{E,F}$ or fail.

- $y \in \mathbb{R}_{>0}$ – minimum imaginary part of points in fiber,
- $d \in \mathbb{Z}_{>0}$ – degree of the first approximation to φ_F in Step 1 below,
- $r \in \mathbb{R}_{\neq 0}$ – real number specified to b bits of precision that defines $Q \in \mathbb{C}/\Lambda$,
- b' – bits of precision when dividing points into $\Gamma_0(N)$ orbits, and
- n – number of trials before we give up and output FAIL.

We compute $P_{E,F,Q}$ using an approach that will always fail if Q is a ramification point. Our algorithm will also fail if any points in the fiber over Q are cusps. This is why we do not allow $r = 0$. One can modify the algorithm to work when Q is an unramified torsion point by using modular symbols and keeping track of images of cusps.

To increase our confidence that we have computed the right point $P_{E,F}$, we often carry out the complete computation with more than one choice of r.

1. **Low precision roots:** Compute all complex double precision roots of the polynomial $\varphi_{F,d} - r$. One way to do this is to use "balanced QR reduction of the companion matrix", as implemented in GSL.[1] Record the roots that correspond to $\tau \in \mathfrak{h}$ with $\mathrm{Im}(\tau) \geq y$.

2. **High precision roots:** Compute B such that if $\mathrm{Im}(\tau) \geq y$, then

$$\left| \sum_{n=B+1}^{\infty} \frac{a_n(F)}{n} \tau^n \right| < 2^{-b},$$

where b is the number of bits of precision of r. Summing the tail end of the series and using that $|a_n| \leq n$ (see [GJP+09, Lem. 2.9]), we find that

$$B = \left\lceil \frac{\log(2^{-(b+1)} \cdot (1 - e^{-2\pi y_1}))}{-2\pi y} \right\rceil$$

works. Next, compute the polynomial $\varphi_{F,B} \in \mathbb{Q}[T]$, and use Newton iteration to refine all roots saved in Step 1 to roots α of $f = \varphi_{F,B} - r \in \mathbb{R}[T]$ such that $|f(\alpha)| < 2^{-b}$. Save those roots that correspond to $\tau \in \mathfrak{h}$ with $\mathrm{Im}(\tau) \geq y$.

3. $\Gamma_0(N)$**-orbits:** Divide the τ's from Step 2 into $\Gamma_0(N)$-equivalence classes, testing equivalence to the chosen bit precision $b' \leq b$, as explained in Section 5.1. It is easy to efficiently compute the modular degree $m_F = \deg(\varphi_F)$ (see [Wat02]). If we find m_F distinct $\Gamma_0(N)$ classes of points, we suspect that we have found the fiber over $[r]$, so we map each element of the fiber to E using φ_E and sum, then apply the elliptic exponential to obtain $P_{E,F}$ to some precision, then output this approximation and terminate. If we find more than m_F distinct classes, there was an error in the choices of precision in our computation, so we output FAIL (and suggest either increasing b or decreasing b').

4. **Try again:** We did not find enough points in the fiber. Systematically replace r by $r + m\Omega_F$, for $m = 1, -1, 2, -2, \ldots$, where Ω_F is the least real period of F, then try again going to Step 1 and including the new points found. If upon trying n choices $r + m\Omega_F$ in a row we find no new points, we output FAIL and terminate the algorithm.

5.1 Determining $\Gamma_0(N)$ equivalency

The field of meromorphic functions invariant under $\Gamma_0(N)$ is generated by $j(z)$ and $j(Nz)$, so if two points z_1 and z_2 in the upper half plane are equivalent under $\Gamma_0(N)$, then z_1 and z_2 are equivalent under $\mathrm{SL}_2(\mathbb{Z})$ and Nz_1 and Nz_2 are also equivalent under $\mathrm{SL}_2(\mathbb{Z})$. Because of singularities in the affine curve defined

[1] GSL is the the GNU scientific library, which is part of Sage [S+11]. Rough timings of GSL for this computation: it takes less than a half second for degree 500, about 5 seconds for degree 1000, about 45 seconds for degree 2000, and several minutes for degree 3000.

by $j(z)$ and $j(Nz)$, the converse is *not* true: for example, $z_1 = (-2+i)/5$ and $z_2 = (2+i)/5$ are equivalent under $\mathrm{SL}_2(\mathbb{Z})$ as are $5z_1$ and $5z_2$, but z_1 and z_2 are not equivalent under $\Gamma_0(5)$. This is why the algorithm we give below must take into account singularities.

Suppose we are given arbitrary z_1 and z_2 in the upper half plane. We first find $g_1, g_2 \in \mathrm{SL}_2(\mathbb{Z})$ such that $w_i = g_i(z_i)$ is the canonical representative for z_i in the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, as explained in [Cre97, §2.14] but using interval arithmetic to avoid rounding errors. If $w_1 \neq w_2$, then z_1 and z_2 are not equivalent under $\mathrm{SL}_2(\mathbb{Z})$, so they cannot be equivalent under $\Gamma_0(N)$. Thus let $w = g_1(z_1) = g_2(z_2)$. The elements of $\mathrm{PSL}_2(\mathbb{Z})$ that send z_1 to z_2 are the finitely many elements $g_2^{-1}Ag_1$, for $A \in \mathrm{Stab}(w)$, so we check whether any $g_2^{-1}Ag_1$ is in $\Gamma_0(N)$. The only elements of the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ with nontrivial stabilizers are $w = i$, with stabilizer generated by $S \in \mathrm{PSL}_2(\mathbb{Z})$ of order 2, and $w = e^{2\pi i/3}$ with stabilizer generated by ST, where T corresponds to $z \mapsto z+1$.

5.2 Data

We implemented the above algorithm in Sage [S^+11][2]. The columns of the tables below are as follows. The columns labeled E and F contain Cremona labels for elliptic curves, and those labeled r_E and r_F contain the corresponding ranks. The column labeled $E(\mathbb{Q})$ gives a choice of generators $P_1, P_2, \ldots$ for the Mordell-Weil group, with r_E points of infinite order listed first, then 0, 1 or 2 torsion points listed with a subscript of their order. The column labeled $P_{E,F}$ contains a rational point close to the numerically computed Chow-Heegner point, represented in terms of the generators P_i from the column labeled $E(\mathbb{Q})$, where P_1 is the first generator, P_2 the second, and so on. The columns labeled m_E and m_F give the modular degrees of E and F. The column labeled ε's contains the local root numbers of $L(E,s)$ at each bad prime. The notes column refers to the notes after the table, which give information about the input parameters needed to compute $P_{E,F}$.

We believe that the values of $P_{E,F}$ are "likely" to be correct, but we emphasize again that *they are not proven correct*. In the table we give an exact point, but the algorithm computes a numerical approximation $\tilde{P}_{E,F}$ to $P_{E,F} \in E(\mathbb{Q})$. We find what we call $P_{E,F}$ in the table by running through several hundred low height points in $E(\mathbb{Q})$ and find the one closest to $\tilde{P}_{E,F}$; in all cases, the coordinates of the point we list are within 10^{-5} of the coordinates of $\tilde{P}_{E,F}$.

The table contains *every* pair E, F of nonisogenous optimal elliptic curves of the same conductor $N \leq 184$ with $r_E = 1$, and *most* (but not all) with $N \leq 250$. It also contains a few additional miscellaneous examples, e.g., with $r_E = 0$ and some of larger conductor with $r_F = 2$. Most rows took only a few seconds to compute, though ones with m_F large in some cases took much longer; the total CPU time to compute the entire table was about 8 hours. Unless otherwise noted, we used $y = 10^{-4}$, $d = 500$, $b' = 20$, and $r = 0.1$ with 53 bits of precision,

[2] See `http://trac.sagemath.org/sage_trac/ticket/11975`.

as in Section 5. We also repeated all computations with at least one additional value of $r \neq 0.1$, and in every case got the same result (usually we used $r = 0.2$).

5.3 Discussion

In the table we always have $2 \mid [E(\mathbb{Q})_{/\,\mathrm{tor}} : \mathbb{Z}P_{E,F}]$. In may be possible to prove this in some cases by using that when $r_{\mathrm{an}}(E) = 1$ then the sign in the functional equation for $L(E,s)$ is -1, so at least one nontrivial Atkin-Lehner involution w_q acts as $+1$ on E, which means that the map $X_0(N) \to E$ factors through $X_0(N) \to X_0(N)/w_q$. Also, there are four cases in which the index $[E(\mathbb{Q})/_{\mathrm{tor}} : \mathbb{Z}P_{E,F}]$ is divisible by a prime $\ell \geq 5$. They are (106b, 106c, $\ell = 11$), (118a, 118d, $\ell = 7$), (121b, 121d, $\ell = 7$), and (158b, 158c, $\ell = 7$). These prime divisors do not appear to have anything to do with the invariants of E and F, individually.

E	ε_p's	r_E	$E(\mathbb{Q})$	m_E	F	r_F	m_F	$P_{E,F}$	Notes
37a	$+$	1	$(0,-1)$	2	37b	0	2	$-6P_1$	
37b	$-$	0	$(8,18)_3$	2	37a	1	2	P_1	
57a	$++$	1	$(2,1)$	4	57c	0	12	$8P_1$	
57a	$++$	1	$(2,1)$	4	57b	0	3	$-8P_1$	
57b	$-+$	0	$(7/4,-11/8)_2,(1,-1)_2$	3	57a	1	4	0	
57b	$-+$	0	$(7/4,-11/8)_2,(1,-1)_2$	3	57c	0	12	0	
57c	$-+$	0	$(2,4)_5$	12	57a	1	4	$3P_1$	
57c	$-+$	0	$(2,4)_5$	12	57b	0	3	P_1	
58a	$++$	1	$(0,-1)$	4	58b	0	4	$8P_1$	
58b	$-+$	0	$(-1,2)_5$	4	58a	1	4	$3P_1$	
77a	$++$	1	$(2,3)$	4	77b	0	20	$24P_1$	(1)
77a	$++$	1	$(2,3)$	4	77c	0	6	$-4P_1$	
89a	$+$	1	$(0,-1)$	2	89b	0	5	$4P_1$	
91a	$++$	1	$(0,0)$	4	91b	1	4	$4P_1$	
91b	$--$	1	$(-1,3),(1,0)_3$	4	91a	1	4	P_2	
92b	$--$	1	$(1,1)$	6	92a	0	2	0	
99a	$++$	1	$(2,0),(-1,0)_2$	4	99b	0	12	$-4P_1$	
99a	$++$	1	$(2,0),(-1,0)_2$	4	99c	0	12	0	
99a	$++$	1	$(2,0),(-1,0)_2$	4	99d	0	6	$2P_1$	
102a	$+++$	1	$(2,-4),(0,0)_2$	8	102b	0	16	$-8P_1$	(1)
102a	$+++$	1	$(2,-4),(0,0)_2$	8	102c	0	24	$32P_1$	
106b	$++$	1	$(2,1)$	8	106a	0	6	$-4P_1$	
106b	$++$	1	$(2,1)$	8	106c	0	48	$-88P_1$	
106b	$++$	1	$(2,1)$	8	106d	0	10	$12P_1$	
112a	$++$	1	$(0,-2),(-2,0)_2$	8	112b	0	4	0	
112a	$++$	1	$(0,-2),(-2,0)_2$	8	112c	0	8	0	
118a	$++$	1	$(0,-1)$	4	118b	0	12	$-8P_1$	(1)
118a	$++$	1	$(0,-1)$	4	118c	0	6	$4P_1$	
118a	$++$	1	$(0,-1)$	4	118d	0	38	$-28P_1$	
121b	$+$	1	$(4,5)$	4	121a	0	6	$4P_1$	
121b	$+$	1	$(4,5)$	4	121c	0	6	$4P_1$	
121b	$+$	1	$(4,5)$	4	121d	0	24	$-28P_1$	(2)
123a	$--$	1	$(-4,1),(-1,4)_5$	20	123b	1	4	0	
123b	$++$	1	$(1,0)$	4	123a	1	20	$4P_1$	
124a	$--$	1	$(-2,1),(0,1)_3$	6	124b	0	6	0	
128a	$+$	1	$(0,1),(-1,0)_2$	4	128b	0	8	0	
128a	$+$	1	$(0,1),(-1,0)_2$	4	128c	0	4	0	
128a	$+$	1	$(0,1),(-1,0)_2$	4	128d	0	8	0	
129a	$++$	1	$(1,-5)$	8	129b	0	15	$-8P_1$	
130a	$+--$	1	$(-6,10),(-1,10)_6$	24	130b	0	8	0	
130a	$+--$	1	$(-6,10),(-1,10)_6$	24	130c	0	80	0	
135a	$++$	1	$(4,-8)$	12	135b	0	36	0	(1)
136a	$--$	1	$(-2,2),(0,0)_2$	8	136b	0	8	0	
138a	$+++$	1	$(1,-2),(-2,1)_2$	8	138b	0	16	$-16P_1$	(1)
138a	$+++$	1	$(1,-2),(-2,1)_2$	8	138c	0	8	$-8P_1$	
141a	$--$	1	$(-3,-5)$	28	141b	0	12	0	
141a	$--$	1	$(-3,-5)$	28	141c	0	6	0	
141a	$--$	1	$(-3,-5)$	28	141d	1	4	0	

E	ε_p's	r_E	$E(\mathbb{Q})$	m_E	F	r_F	m_F	$P_{E,F}$	Notes
141a	- -	1	$(-3,-5)$	28	141e	0	12	0	
141d	++	1	$(0,-1)$	4	141a	1	28	$-12P_1$	
141d	++	1	$(0,-1)$	4	141b	0	12	$4P_1$	
141d	++	1	$(0,-1)$	4	141c	0	6	$4P_1$	
141d	++	1	$(0,-1)$	4	141e	0	12	$4P_1$	
142a	- -	1	$(1,1)$	36	142b	1	4	0	
142a	- -	1	$(1,1)$	36	142c	0	9	0	
142a	$--$	1	$(1,1)$	36	142d	0	4	0	
142a	$--$	1	$(1,1)$	36	142e	0	324	0	(2)
142b	++	1	$(-1,0)$	4	142a	1	36	$4P_1$	(1)
142b	++	1	$(-1,0)$	4	142c	0	9	$-4P_1$	
142b	++	1	$(-1,0)$	4	142d	0	4	$4P_1$	
142b	++	1	$(-1,0)$	4	142e	0	324	$8P_1$	(2)
152a	++	1	$(-1,-2)$	8	152b	0	8	0	
153a	++	1	$(0,1)$	8	153b	1	16	$8P_1$	
153a	++	1	$(0,1)$	8	153c	0	8	$8P_1$	
153a	++	1	$(0,1)$	8	153d	0	24	0	
153b	$--$	1	$(5,-14)$	16	153a	1	8	0	
153b	$--$	1	$(5,-14)$	16	153d	0	24	0	
154a	$+++$	1	$(5,3),(-6,3)_2$	24	154b	0	24	$-24P_1$	
154a	$+++$	1	$(5,3),(-6,3)_2$	24	154c	0	16	$16P_1$	
155a	$--$	1	$(5/4,31/8),(0,2)_5$	20	155b	0	8	0	
155a	$--$	1	$(5/4,31/8),(0,2)_5$	20	155c	1	4	0	
155c	++	1	$(1,-1)$	4	155a	1	20	$-12P_1$	
155c	++	1	$(1,-1)$	4	155b	0	8	$4P_1$	
156a	$-+-$	1	$(1,1),(2,0)_2$	12	156b	0	12	0	(1)
158a	$--$	1	$(-1,-4)$	32	158b	1	8	0	
158a	$--$	1	$(-1,-4)$	32	158c	0	48	0	(1)
158a	$--$	1	$(-1,-4)$	32	158d	0	40	0	
158a	$--$	1	$(-1,-4)$	32	158e	0	6	0	
158b	++	1	$(0,-1)$	8	158a	1	32	$-8P_1$	
158b	++	1	$(0,-1)$	8	158c	0	48	$-56P_1$	(1)
158b	++	1	$(0,-1)$	8	158d	0	40	0	
158b	++	1	$(0,-1)$	8	158e	0	6	$-8P_1$	
160a	++	1	$(2,-2),(1,0)_2$	8	160b	0	8	0	
162a	++	1	$(-2,4),(1,1)_3$	12	162b	0	6	0	
162a	++	1	$(-2,4),(1,1)_3$	12	162c	0	6	0	
162a	++	1	$(-2,4),(1,1)_3$	12	162d	0	12	0	
170a	$+--$	1	$(0,2),(1,-1)_2$	16	170d	0	12	0	
170a	$+--$	1	$(0,2),(1,-1)_2$	16	170e	0	20	0	
171b	$--$	1	$(2,-5)$	8	171a	0	12	0	
171b	$--$	1	$(2,-5)$	8	171c	0	96	0	(1)
171b	$--$	1	$(2,-5)$	8	171d	0	32	0	
175a	$--$	1	$(2,-3)$	8	175b	1	16	0	(1)
175a	$--$	1	$(2,-3)$	8	175c	0	40	0	(1)
175b	++	1	$(-3,12)$	16	175a	1	8	$16P_1$	
175b	++	1	$(-3,12)$	16	175c	0	40	$16P_1$	(1)
176c	$--$	1	$(1,-2)$	8	176b	0	8	0	(1)

E	ε_p's	r_E	$E(\mathbb{Q})$	m_E	F	r_F	m_F	$P_{E,F}$	Notes
176c	$--$	1	$(1,-2)$	8	176a	0	16	0	
176c	$--$	1	$(1,-2)$	8	176b	0	8	0	(1)
184a	$--$	1	$(0,1)$	8	184c	0	12	0	
184a	$--$	1	$(0,1)$	8	184d	0	24	0	
184b	$++$	1	$(2,-1)$	8	184a	1	8	0	
184b	$++$	1	$(2,-1)$	8	184c	0	12	0	
184b	$++$	1	$(2,-1)$	8	184d	0	24	0	
185a	$++$	1	$(4,-13)$	48	185b	1	8	$8P_1$	
185a	$++$	1	$(4,-13)$	48	185c	1	6	$24P_1$	
185b	$--$	1	$(0,2)$	8	185c	1	6	0	
185c	$++$	1	$(-5/4,3/8),(-1,0)_2$	6	185b	1	8	$2P_1$	
189a	$++$	1	$(-1,-2)$	12	189b	1	12	$-12P_1$	
189a	$++$	1	$(-1,-2)$	12	189c	0	12	$12P_1$	
189b	$--$	1	$(-3,9),(3,0)_3$	12	189a	1	12	0	
189b	$--$	1	$(-3,9),(3,0)_3$	12	189c	0	12	0	
190a	$-+-$	1	$(13,-47)$	88	190b	1	8	0	
190a	$-+-$	1	$(13,-47)$	88	190c	0	24	0	(1)
190b	$+++$	1	$(1,2)$	8	190c	0	24	$16P_1$	(1)
192a	$++$	1	$(3,2),(-1,0)_2$	8	192b	0	8	0	
192a	$++$	1	$(3,2),(-1,0)_2$	8	192c	0	8	0	
192a	$++$	1	$(3,2),(-1,0)_2$	8	192d	0	8	0	
196a	$--$	1	$(0,-1)$	6	196b	0	42	0	(1)
198a	$+--$	1	$(-1,-4),(-4,2)_2$	32	198b	0	32	0	(1)
198a	$+--$	1	$(-1,-4),(-4,2)_2$	32	198c	0	32	0	
198a	$+--$	1	$(-1,-4),(-4,2)_2$	32	198d	0	32	0	(1)
198a	$+--$	1	$(-1,-4),(-4,2)_2$	32	198e	0	160	0	(1)
200b	$--$	1	$(-1,1),(-2,0)_2$	8	200c	0	24	0	
200b	$--$	1	$(-1,1),(-2,0)_2$	8	200d	0	40	0	(1)
200b	$--$	1	$(-1,1),(-2,0)_2$	8	200e	0	24	0	
201a	$++$	1	$(1,-2)$	12	201b	1	12	$4P_1$	
201b	$--$	1	$(-1,2)$	12	201a	1	12	0	
201c	$++$	1	$(16,-7)$	60	201a	1	12	$-24P_1$	
201c	$++$	1	$(16,-7)$	60	201b	1	12	$8P_1$	
203b	$--$	1	$(2,-5)$	8	203a	0	48	0	
203b	$--$	1	$(2,-5)$	8	203c	0	12	0	
205a	$--$	1	$(-1,8),(2,1)_4$	12	205b	0	16	0	
205a	$--$	1	$(-1,8),(2,1)_4$	12	205c	0	8	0	
208a	$--$	1	$(4,-8)$	16	208c	0	12	0	
208a	$--$	1	$(4,-8)$	16	208d	0	48	0	(1)
208b	$++$	1	$(4,4)$	16	208a	1	16	0	(1)
208b	$++$	1	$(4,4)$	16	208c	0	12	0	
208b	$++$	1	$(4,4)$	16	208d	0	48	0	(1)
212a	$--$	1	$(2,2)$	12	212b	0	21	0	
214a	$--$	1	$(0,-4)$	28	214b	1	12	0	(1)
214a	$--$	1	$(0,-4)$	28	214d	0	12	0	
214b	$++$	1	$(0,0)$	12	214a	1	28	$-8P_1$	(1)
214b	$++$	1	$(0,0)$	12	214d	0	12	$-4P_1$	

E	ε_p's	r_E	$E(\mathbb{Q})$	m_E	F	r_F	m_F	$P_{E,F}$	Notes
214c	$++$	1	$(11,10)$	60	214a	1	28	$-4P_1$	(1)
214c	$++$	1	$(11,10)$	60	214d	0	12	$16P_1$	
214c	$++$	1	$(11,10)$	60	214b	1	12	$12P_1$	(1)
216a	$++$	1	$(-2,-6)$	24	216b	0	24	0	
219a	$++$	1	$(2,-1)$	12	219c	1	60	$-12P_1$	(1)
219a	$++$	1	$(2,-1)$	12	219b	1	12	$-4P_1$	
216a	$++$	1	$(-2,-6)$	24	216d	0	72	0	
219b	$--$	1	$(-3/4,-1/8),(0,1)_3$	12	219a	1	12	0	
219b	$--$	1	$(-3/4,-1/8),(0,1)_3$	12	219c	1	60	0	(1)
219c	$++$	1	$(-6,7),(10,-5)_2$	60	219a	1	12	$-12P_1$	
219c	$++$	1	$(-6,7),(10,-5)_2$	60	219b	1	12	$4P_1$	
220a	$--+$	1	$(-7,11),(15,55)_6$	36	220b	0	12	0	
224a	$++$	1	$(1,2),(0,0)_2$	8	224b	0	8	0	
225a	$++$	1	$(1,1)$	8	225b	0	40	0	(1)
225e	$--$	1	$(-5,22)$	48	225a	1	8	0	(1)
225e	$--$	1	$(-5,22)$	48	225b	0	40	0	(1)
228b	$-+-$	1	$(3,6)$	24	228a	0	18	0	
232a	$++$	1	$(2,-4)$	16	232b	0	16	0	
234c	$+++$	1	$(1,-2),(-2,1)_2$	16	234b	0	48	0	(1)
234c	$+++$	1	$(1,-2),(-2,1)_2$	16	234e	0	20	0	(1)
235a	$--$	1	$(-2,3)$	12	235c	0	18	0	(1)
236a	$--$	1	$(1,-1)$	6	236b	0	14	0	
238a	$--+$	1	$(24,100),(-8,4)_2$	112	238b	1	8	0	(1)
238a	$--+$	1	$(24,100),(-8,4)_2$	112	238c	0	16	0	(1)
238a	$--+$	1	$(24,100),(-8,4)_2$	112	238d	0	16	0	(1)
238b	$+++$	1	$(1,1),(0,0)_2$	8	238a	1	112	$12P_1$	(1)
238b	$+++$	1	$(1,1),(0,0)_2$	8	238c	0	16	$-4P_1$	(1)
238b	$+++$	1	$(1,1),(0,0)_2$	8	238d	0	16	$4P_1$	(1)
240c	$+++$	1	$(1,2),(0,0)_2$	16	240a	0	16	0	
240c	$+++$	1	$(1,2),(0,0)_2$	16	240d	0	16	0	(1)
243a	$+$	1	$(1,0)$	6	243b	0	9	0	(1)
245a	$--$	1	$(7,17)$	48	245c	1	32	0	
246d	$+++$	1	$(3,-6),(4,-2)_2$	48	246a	0	84	$24P_1$	(1)
446a	$++$	1	$(4,-6)$	24	446d	2	88	0	(2)
446b	$--$	1	$(5,-10)$	56	446d	2	88	0	(2)
446d	$+-$	2	-	88	446a	1	12	0	(1)
446d	$+-$	2	-	88	446b	1	56	0	(1)
681a	$++$	1	$(4,4)$	32	681c	2	96	$-24P_1$	(2)

Notes:

(1) We used $y = 10^{-5}$ and $d = 1500$, which takes a few minutes.

(2) We used $y = \frac{1}{2} \cdot 10^{-5}$ and $d = 3000$, which takes over an hour.

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