Harvard Math 129: Algebraic Number Theory
Homework Assignment 10

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Due: Thursday, May 5, 2005

The problems have equal point value, and multi-part problems are of the same value. You are encouraged to look at books for inspiration about how to solve any of the following problems.

1. Let K be a number field. Prove that there is a finite set S of primes of K such that
$$\mathcal{O}_{K,S} = \{a \in K^* : \mathrm{ord}_{\mathfrak{p}}(a\mathcal{O}_K) \geq 0 \text{ all } \mathfrak{p} \not\in S\} \cup \{0\}$$
is a prinicipal ideal domain. The condition $\mathrm{ord}_{\mathfrak{p}}(a\mathcal{O}_K) \geq 0$ means that in the prime ideal factorization of the fractional ideal $a\mathcal{O}_K$, we have that $\mathfrak{p}$ occurs to a nonnegative power. [See the notes for hints.]

2. Let $a \in K$ and n a positive integer. Prove that $L = K(a^{1/n})$ is unramified outside the primes that divide n and the norm of a. This means that if $\mathfrak{p}$ is a prime of $\mathcal{O}_K$, and $\mathfrak{p}$ is coprime to $n\,\mathrm{Norm}_{L/K}(a)\mathcal{O}_K$, then the prime factorization of $\mathfrak{p}\mathcal{O}_L$ involves no primes with exponent bigger than 1.

3. Write down a proof of Hilbert's Theorem 90, formulated as the statement that for any number field K, we have
$$\mathrm{H}^1(K, \overline{K}^*) = 0.$$