Explicit Modular Families of K3 Surfaces

- joint work with Adrian Clingher

Themes: - CALABI-YAU MANIFOLDS AND MODULAR FORMS

- STRING DUALITY ⇒ MATHEMATICAL CONJECTURES

- NORMAL FORM FOR ELLIPTIC CURVES, GENUS TWO CURVES, AND CERTAIN K3 SURFACES

- HODGE-THEORETIC CORRESPONDENCE ⇒ GEOMETRIC CORRESPONDENCE (THE HODGE CONJECTURE)

- EXPLICIT PARAMETRIZATION OF (RATIONAL) MODULI SPACES

- ROLE OF SPECIAL ELLIPTIC FIBRATIONS WITH SECTION ON K3 SURFACES, SUGGESTED BY "D=8 F-THEORY/HETEROTIC STRING DUALITY"

Calabi-Yau Manifolds

- trivial canonical bundle : $K_X \cong \mathcal{O}_X$

Dimension 1 : · elliptic curves

Dimension 2 : · abelian surfaces
(eg. $\mathrm{Jac}(C)$, $\mathrm{genus}(C) = 2$;
or $E_1 \times E_2$, E_i elliptic)

· K3 surfaces (e.g. quartic $\subset \mathbb{P}^3$)

Dimension 3 : · Calabi-Yau threefolds

Topology of K3 surfaces :

- simply connected
- $H_2(K3, \mathbb{Z}) \cong H \oplus H \oplus H \oplus E_8 \oplus E_8$

where H = standard hyperbolic lattice of rank 2 w/ intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and E_8 = negative definite lattice assoc. to E_8 Dynkin diagram. (even, unimodular)

Prototype example: elliptic curves

- Legendre <u>normal form</u> for genus 1 curve

$$E_\lambda : y^2 = x(x-1)(x-\lambda) \qquad [\text{affine equation}]$$

- <u>Periods</u> given by elliptic integrals

$$\omega_i = \int_{\gamma_i} \frac{dx}{y} \quad , \ \gamma_1, \gamma_2 \text{ basis of } H_1(E_\lambda; \mathbb{Z})$$

$$\omega_i(\lambda) \qquad\qquad \tau(\lambda) := \frac{\omega_2(\lambda)}{\omega_1(\lambda)}$$

$$\begin{tikzcd}
 & \tau(\lambda) \in \mathfrak{h} \arrow[dr, "J(\tau)"] & \\
\lambda \in \mathbb{CP}^1 \setminus \{0,1,\infty\} \arrow[ur, mapsto] \arrow[ur, Rightarrow, shift left=2] \arrow[rr, "j(\lambda)"'] & & \mathbb{C} = PSL(2,\mathbb{Z}) \backslash \mathfrak{h}
\end{tikzcd}$$

$J(\tau)$ elliptic modular function

$$\lambda(\tau) = \frac{-\theta\begin{bmatrix}1\\0\end{bmatrix}(0;\tau)^4}{\theta\begin{bmatrix}0\\1\end{bmatrix}(0;\tau)^4} \qquad j(\lambda) = \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$

Moduli space: $\mathcal{M}_E = \mathbb{C} = PSL(2,\mathbb{Z}) \backslash \mathfrak{h}$

arithmetic quotient of period domain ($\mathfrak{h}$)

Algebraic relation between variable in normal form equation (λ) and modular parameter (J)

/

Want to study certain moduli spaces $\mathcal{M}_{K3}^{N}$ of K3 surfaces, normal forms of these K3 surfaces, and derive explicit algebraic relation between variables in normal form equation and modular parameters

K3 Surfaces Polarized by $H \oplus E_8 \oplus E_7$

N = even lattice $H \oplus \underbrace{E_8 \oplus E_7}$

std. hyp. latt. of rank 2

neg. definite lattices assoc. to E_8 and E_7 Dynkin diagrams

rank$(N) = 17$
signature$(N) = (1, 16)$

X K3 surface

$NS(X)$ = Néron-Severi lattice of X

- N-polarization on X is a primitive lattice embedding $i : N \hookrightarrow NS(X)$ such that the image $i(N)$ contains a pseudo-ample class
- Two N-polarized K3 surfaces (X_1, i_1) and (X_2, i_2) are isomorphic if there exists an analytic isomorphism $\alpha : X_1 \to X_2$ such that $\alpha^* \circ i_2 = i_1$

Can construct explicit examples of N-polarized K3 surfaces by resolving singularities of certain projective quartic surfaces

For $(\alpha, \beta, \gamma) \in \mathbb{C}^3$, consider $Q(\alpha,\beta,\gamma) \subset \mathbb{P}^3$:
$[x, y, z, w]$

$$y^2 z w - 4x^3 z + 3\alpha x z w^2 + \beta z w^3 + \gamma x z^2 w - \frac{1}{2}(z^2 w^2 + w^4) = 0$$

"EXTENDED" INOSE FAMILY

$$\left.\begin{array}{l} P_1 = [0,1,0,0] \\ P_2 = [0,0,1,0] \end{array}\right\} \text{ generic singular loci of } Q(\alpha,\beta,\gamma)$$

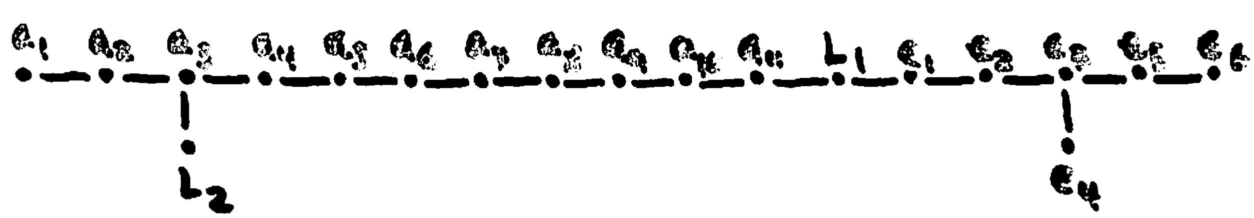
P_1 : always rational double pt of type A_{11}

P_2 : $\gamma \neq 0 \Rightarrow$ type A_5 ; $\gamma = 0 \Rightarrow$ type E_6

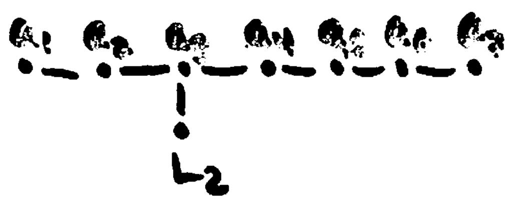
L_1 = line $x = w = 0$, passing through P_1 and P_2
L_2 = line $z = w = 0$, passing through P_1

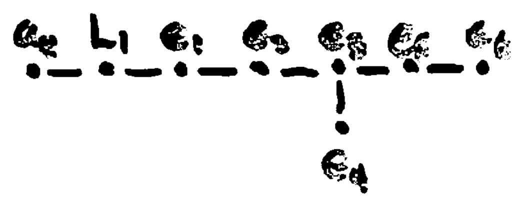
The surface $X(\alpha,\beta,\gamma)$ obtained as the minimal resolution of $Q(\alpha,\beta,\gamma)$ is a K3 surface endowed with a canonical N-polarization

Case $\gamma = 0$: Resolve P_1 and P_2 $\Rightarrow$ special configuration of rational curves :



E_8





E_8

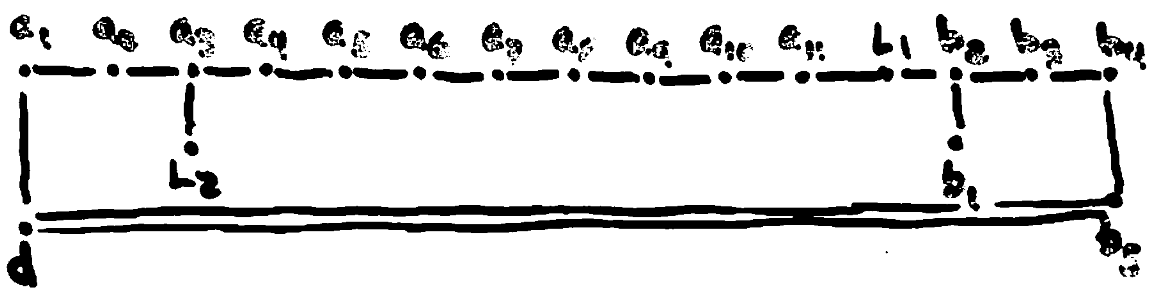
$\gamma = 0$ corresponds to enhancement from

$H \oplus E_8 \oplus E_7$ to $H \oplus E_8 \oplus E_8$ lattice polarization

What is the H? $\quad H = \langle a_9, f \rangle$

$$f = a_8 + 2a_7 + 3a_6 + 4a_5 + 5a_4 + 6a_3 + 3L_2 + 4a_2 + 2a_1$$
$$= a_{10} + 2a_{11} + 3L_1 + 4e_1 + 5e_2 + 6e_3 + 3e_4 + 2e_5 + e_6$$

<u>Case $\gamma \neq 0$</u>: Special configuration upon resolution



<u>The curve d</u> is the intersection of the hyperplane

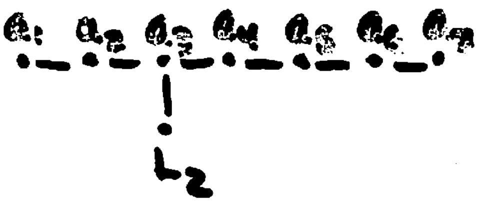
$2\gamma x = w$ with the cubic hypersurface

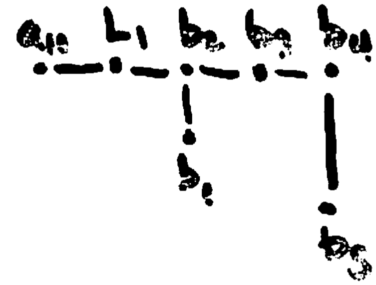
$4\gamma^4 x^3 + (2 - 6\alpha\gamma^2 - 4\beta\gamma^3)x^2 z - \gamma y^2 z = 0$

in $Q(\alpha,\beta,\gamma)$ resolves to a rational curve

d in $X(\alpha,\beta,\gamma)$

E_8





E_7

with $H = \langle a_9, f \rangle$ again.

The extended Inose family $X(\alpha,\beta,\gamma)$ is a normal form for N-polarized K3 surfaces:

Given any N-polarized K3 surface (X, i), one can choose a triple $(\alpha,\beta,\gamma) \in \mathbb{C}^3$ such that $X(\alpha,\beta,\gamma)$ with its canonical N-polarization is isomorphic to (X, i)

Discriminant locus of family of quartics $Q(\alpha,\beta,\gamma)$

- one component : locus $\gamma = 0$

  N-polarization $\Rightarrow$ $H \oplus E_8 \oplus E_8$-polariz.

- second component where $Q(\alpha,\beta,\gamma)$ becomes singular away from P_1 and P_2

  N-polarization $\Rightarrow$ $H \oplus E_8 \oplus E_7 \oplus A_1$-polariz

- intersection of these two components:

  quartics $Q(\alpha,\beta,0)$ with $\alpha^3 = (\beta+1)^2$

  or $\alpha^3 = (\beta-1)^2$

  Surfaces in these two distinct families have canonical $H \oplus E_8 \oplus E_8 \oplus A_1$-polarizations

Hodge Theory and Moduli of N-Polarized K3 Surfaces

$\mathcal{M}_{K3}^{N}$

glue together local deformation spaces $\rightsquigarrow$ coarse moduli space for isom. classes of N-polarized K3 surfaces

/ $\mathcal{M}_{K3}^{N}$ is a quasi-projective analytic space of complex dimension 3

Via period map and version of Global Torelli Thm use Hodge Theory to analyze structure of $\mathcal{M}_{K3}^{N}$

- Up to overall isometry, there exists a unique primitive embedding j of N into the K3 lattice, $L = H \oplus H \oplus H \oplus E_8 \oplus E_8$
- Fix such a lattice embedding. Denote by T the orthogonal complement of its image

$$\Rightarrow \Omega = \left\{ \omega \in \mathbb{P}(T \otimes \mathbb{C}) \;\middle|\; (\omega, \omega) = 0,\ (\omega, \bar{\omega}) > 0 \right\}$$

period domain

$$\mathrm{per} : \mathcal{M}^{N}_{K3} \xrightarrow{\cong} O(T)\backslash\Omega$$

period map

classifying space of N-polarized Hodge structures

$T \cong H \oplus H \oplus A_1$

Pick an integral basis $\{p_1, p_2, q_1, q_2, r\}$ for T

Intersection matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}$$

Hodge-Riemann bilinear relations $\Rightarrow$ can realize every period $\omega \in \Omega$ uniquely as

$$\omega(\tau, u, z) = (\tau, 1, u, z^2 - \tau u, z)$$

with $\tau, u, z \in \mathbb{C}$ s.t. $\tau_2 \cdot u_2 > z_2^2$

subscripts indicate imaginary parts

Ω has two connected components which get <u>interchanged</u> by complex conjugation : $\Omega_0, \overline{\Omega_0}$

$$Z := \begin{pmatrix} \tau & z \\ z & u \end{pmatrix} \longmapsto \omega(\tau, u, z)$$

$$\mathbb{H}_2 \xrightarrow{\cong} \Omega_0$$

where $\mathbb{H}_2$ is the classical Siegel upper half space of genus two:

$$\mathbb{H}_2 := \left\{ Z \;\middle|\; \tau_2 u_2 > z_2^2 ,\ \tau_2 > 0 \right\}$$

Furthermore, can see action of $\mathcal{O}(T)$:

$$Sp_4(\mathbb{Z}) / \{\pm I_4\} \xrightarrow{\cong} \mathcal{O}^+(T) / \{\pm id\}$$

where $\mathcal{O}^+(T) := \mathcal{O}^+(T, \mathbb{R}) \cap \mathcal{O}(T)$

and $\mathcal{O}^+(T, \mathbb{R})$ is the (index two) subgroup of $\mathcal{O}(T, \mathbb{R})$ fixing the component Ω_0

$$\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix} \longmapsto \begin{pmatrix} -b_4c_1 + a_3d_2 & \cdots & b_3c_1 - b_4c_2 - a_3d_1 + a_4d_2 \\ c_3d_2 - c_1d_4 & \cdots & -c_3d_1 + c_4d_2 + c_1d_3 - c_2d_4 \\ -b_4c_1 + a_3d_2 & \cdots & b_3c_1 - b_4c_2 - a_3d_1 + a_4d_2 \\ a_3b_2 - a_1b_4 & \cdots & -a_3b_1 + a_4b_2 + a_1b_3 - a_2b_4 \\ -b_4c_3 + a_3d_4 & \cdots & b_3c_3 - b_4c_4 - a_3d_3 + a_4d_4 \end{pmatrix}$$

in $Sp_4(\mathbb{Z})$ as isometry in $\mathcal{O}^+(T)$

We obtain an isomorphism

$$\Gamma_2 \backslash \mathbb{H}_2 \cong O^+(T) \backslash \Omega_0 \cong O(T) \backslash \Omega$$

and so the period isomorphism analytically identifies the moduli space $\mathcal{M}^N_{K3}$ with the standard <u>Siegel</u> <u>modular</u> <u>threefold</u>

$$\mathcal{F}_2 = \Gamma_2 \backslash \mathbb{H}_2$$

- Noncompact
- Highly singular

$\mathrm{Sing}(\mathcal{F}_2)$ = images, under $\mathbb{H}_2 \to \Gamma_2 \backslash \mathbb{H}_2$, of points in $\mathbb{H}_2$ whose associated periods $\omega(\tau, u, z)$ are orthogonal to roots of the rank 5 lattice T

$T \cong H \oplus H \oplus A_1 \Rightarrow$ roots of T form <u>two</u> <u>distinct</u> <u>orbits</u> under $O(T)$-action, distinguished by lattice type of orthogonal complement $\delta^\perp$ for each root δ

$$\delta^\perp = H \oplus H \quad \text{or} \quad H \oplus (-A_1) \oplus A_1$$

So $\mathrm{Sing}(\mathcal{F}_2)$ has two connected components ~ the two <u>Humbert</u> <u>surfaces</u> $\mathcal{H}_1$ and $\mathcal{H}_4$

$\mathcal{H}_1$ = image of the divisor of $\mathbb{H}_2$ assoc. to $z = 0$

$\mathcal{H}_4$ = image of the divisor of $\mathbb{H}_2$ assoc. to $\tau = u$

Each is isomorphic, as an analytic space, to the Hilbert modular surface

$$(PSL(2,\mathbb{Z}) \times PSL(2,\mathbb{Z})) \rtimes \mathbb{Z}/2\mathbb{Z} \backslash \mathfrak{h} \times \mathfrak{h}$$

Meaning on $\mathcal{M}^{N}_{K3}$?

$\mathcal{H}_1$ = locus of N-polarized K3 surfaces (X, i) for which the lattice polarizations are the restrictions of $H \oplus E_8 \oplus E_8$-polarizations

$\mathcal{H}_4$ = ... $H \oplus E_8 \oplus E_7 \oplus A_1$-polarizations

Siegel <u>modular</u> forms ?.

Eisenstein series

$$\mathcal{E}_{2k}(Z) = \sum_{(C,D)} \det(CZ+D)^{-2k}, \quad k>1$$

equiv. classes of coprime 2×2 symmetric matrix pairs

Genus two cusp forms via genus two thetas:

$\theta_m(Z)$

$$\mathcal{C}_{10}(Z) = 2^{-12} \cdot \prod_{m \text{ even}} \theta_m(Z)^2$$

$$\mathcal{C}_{12}(Z) = 2^{-15} \cdot \sum_{\star} \left(\theta_{m_1}(Z)\theta_{m_2}(Z)\theta_{m_3}(Z) \cdot \ldots \cdot \theta_{m_6}(Z)\right)^4$$

$\star$: sum taken over the complements of the 15 syzygous (Göpel) quadruples (m_1, m_2, m_3, m_4)

$$\mathcal{C}_{35}(Z) = 2^{-37} \cdot 5^{-3} \cdot \left(\prod_{m \text{ even}} \theta_m(Z)\right) \cdot \left(\sum_{\circ} \pm \left(\theta_{m_1}(Z)\theta_{m_2}(Z)\theta_{m_3}(Z)\right)^{20}\right)$$

$\circ$: sum taken over the 60 asyzygous triples (m_1, m_2, m_3) of even characteristics

[Two more "cusp forms" $\mathcal{C}_5$ and $\mathcal{C}_{30}$ s.t.

$\mathcal{C}_5^2 = \mathcal{C}_{10}$ and $\mathcal{C}_5\mathcal{C}_{30} = \mathcal{C}_{35}$]

Igusa's structure theorem:

The graded ring of genus two Siegel modular forms is generated by $\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}$, and $\mathcal{C}_{35}$ and is isomorphic to:

$$\mathbb{C}[\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}, \mathcal{C}_{35}] \Big/ \left(\mathcal{C}_{35}^2 = P(\mathcal{E}_4, \mathcal{E}_6, \mathcal{C}_{10}, \mathcal{C}_{12}) \right)$$

where P is a polynomial in which each monomial has weighted degree 70.

$\mathrm{Sing}(\mathcal{F}_2)$ = vanishing divisor of the Siegel cusp form $\mathcal{C}_{35}$

$\mathcal{H}_1 : \{\mathcal{C}_5 = 0\}$, $\mathcal{H}_4 : \{\mathcal{C}_{30} = 0\}$

<u>Thm</u>: There exists a Hodge-theoretic correspondence $(A, \Sigma) \rightsquigarrow (X, i)$ associating bijectively to every N-polarized K3 surface (X, i) a unique polarized abelian surface (A, Σ). This correspondence underlies an analytic identification $\mathcal{A}_2(\mathbb{C}) \longleftrightarrow \mathcal{M}^N_{K3}$ between the two coarse moduli spaces

Principal polarization $\mathcal{L}$ on a complex abelian surface A can be of two types:

(i) $\mathcal{L} = \mathcal{O}_A(E_1 + E_2)$ where E_1 and E_2 are smooth elliptic curves. In this case A splits canonically as a cartesian product $E_1 \times E_2$

(ii) $\mathcal{L} = \mathcal{O}_A(C)$ where C is a smooth genus two curve. In this case A is canonically identified with the Jacobian $\mathrm{Jac}(C)$, with the divisor C being given by the image of the Abel-Jacobi map

Let's refine the Hodge-theoretic correspondence between $(A, \mathcal{L})$ and (X, ι) by analyzing each case separately

[redacted]

Case (i):

$(A, \mathcal{L})$ corresponds to a Siegel period point

$\varepsilon \in \mathfrak{H}_1$ (i.e. $C_5(\varepsilon) = 0$)

The unordered pair (E_1, E_2) of elliptic curves determines the isomorphism class of $(A, \mathcal{L})$

Explicit computation of J-invariants of the two elliptic curves in terms of ε:

$$J(E_1) \cdot J(E_2) = 2^{-12} \cdot 3^{-6} \cdot \left(\frac{\mathcal{E}_4^3}{C_{12}}\right)(\varepsilon)$$

$$J(E_1) + J(E_2) = 2^{-12} \cdot 3^{-6} \cdot \left(\frac{\mathcal{E}_4^3 - \mathcal{E}_6^2}{C_{12}}\right)(\varepsilon) + 1$$

Q: What are $J(E_1)$ and $J(E_2)$ for E_1, E_2 corresponding to $X(\alpha, \beta, \gamma)$?

[redacted]

Case (ii):

$(A, \mathcal{L})$ corresponds to $\varepsilon \in \underbrace{\mathcal{J}_2 \setminus \mathcal{H}_1}$

- coarse moduli space for Jacobian surfaces with canonical principal polarization

$(\mathrm{Jac}(C), \Theta_{\mathrm{Jac}(C)}(C))$

C $\equiv$ • moduli space for genus 2 curves

smooth genus 2 curve

bicanonical map $\Rightarrow$ canonical hyperelliptic structure

$$C: \quad y^2 = (x-\lambda_1)(x-\lambda_2)\cdot \ldots \cdot (x-\lambda_6)$$

Convention: $(ij) = \lambda_i - \lambda_j$

Igusa-Clebsch parameters: $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$

$$\mathcal{A} := \sum_{\text{fifteen}} (12)^2(34)^2(56)^2$$

$$\mathcal{B} := \sum_{\text{ten}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2$$

$$\mathcal{C} := \sum_{\text{sixty}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2$$

$$\mathcal{D} := \prod_{i<j} (ij)^2$$

Igusa-Clebsch invariant of curve C:

$$[\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}] \in \mathbb{WP}^3(2,4,6,10)$$

The affine variety $\{[a,b,c,d] \in \mathbb{WP}^3(2,4,6,10) \mid d \neq 0\}$ is a moduli space for complex genus two curves

$$\left(\Gamma_2 \backslash \mathbb{H}_2\right) \setminus \mathcal{H}_1 \longrightarrow \mathbb{WP}^3(2,4,6,10)$$

$$[\varepsilon] \mapsto \Bigg[-2^3 \cdot 3 \left(\frac{\mathcal{C}_{12}}{\mathcal{C}_{10}}\right)(\varepsilon),\ 2^2 \cdot \mathcal{E}_4(\varepsilon),$$

$$-2^3 \cdot 3^{-1} \left(\frac{2^2 \cdot 3 \cdot \mathcal{E}_4 \mathcal{E}_{12} - \mathcal{E}_6 \mathcal{C}_{10}}{\mathcal{C}_{10}}\right)(\varepsilon),\ -2^{14} \cdot \mathcal{C}_{10}(\varepsilon)\Bigg]$$

[Note: $\mathcal{C}_{10} = \mathcal{C}_5^{\,2}$, so well-defined]

<u>Thm</u>: The Hodge theoretic correspondence divides into two subcases:

(i) a bijective correspondence

$\{E_1, E_2\} \rightsquigarrow (X, i)$ relating unordered pairs of smooth complex elliptic curves to K3 surfaces polarized by the rank 18 lattice $H \oplus E_8 \oplus E_8$

(ii) a bijective correspondence

$C \rightsquigarrow (X, i)$ relating complex curves of genus two to N-polarized K3 surfaces for which the lattice polarization <u>cannot</u> be extended to an $H \oplus E_8 \oplus E_8$ polarization

Corresponding abelian surfaces:

(i) $(E_1 \times E_2,\ \mathcal{O}_{E_1 \times E_2}(E_1 + E_2))$

(ii) $(\mathrm{Jac}(C),\ \mathcal{O}_{\mathrm{Jac}(C)}(C))$

Q: Is there a purely geometric construction underlying the correspondence between (X, i) and (A, Z) in each case?

What are, explicitly:

- the elliptic curves E_1, E_2 corresponding to $X(\alpha, \beta, 0)$ (i.e. their J-invariants)?
- the Igusa–Clebsch invariants of the genus two curve C corresponding to $X(\alpha, \beta, \gamma)$ ($\gamma \neq 0$)?

Note: The Hodge Conjecture suggests the answer to the general question is "yes":

$$Z \subset A \times X$$

existence of algebraic correspondence

$$[Z] \in H^4(A \times X, \mathbb{Q})$$

Explicit answers to above questions should (and do) provide explicit construction of this algebraic correspondence

Thm: Let (X, i) be an N-polarized K3 surface

Then:

(a) The surface X carries a canonical involution $i_{SI} : X \to X$ such that $i_{SI}^*(\omega) = \omega$ for any holomorphic two-form ω of X.

(b) The minimal resolution Y of X / i_{SI} is a Kummer surface endowed with a canonical Kummer structure and a canonical (4)-polarization

(c) The Kummer structure of (b) canonically identifies Y with the Kummer surface $\mathrm{Km}(A)$ associated to the canonically defined associated principally polarized complex abelian surface $(A, \mathcal{L})$

(d) The transformation

$(X, i) \dashrightarrow (A, \mathcal{L})$ induces naturally the Hodge isometry described above.

Thm: Explicit formulas for the modular invariants of $X(\alpha,\beta,\gamma)$ are given as follows:

(i) The K3 surface $X(\alpha,\beta,0)$ corresponds to a pair of elliptic curves E_1, E_2 such that

$$J(E_1)\cdot J(E_2) = \alpha^3$$

$$J(E_1) + J(E_2) = \alpha^3 - \beta^2 + 1$$

(ii) The K3 surface $X(\alpha,\beta,\gamma)$ corresponds to a genus two curve C whose Igusa-Clebsch invariants satisfy:

$$\gamma^6 = 2^{13}\cdot 3^5 \cdot \frac{\mathcal{D}}{\mathcal{A}^5}$$

$$\alpha\cdot\gamma^2 = 2^4 \cdot \frac{\mathcal{B}}{\mathcal{A}^2}$$

$$\beta\cdot\gamma^3 = 2^6 \, \frac{(3\mathcal{C} - \mathcal{A}\cdot\mathcal{B})}{\mathcal{A}^3}$$

where we have parametrized the affine open given by $\mathcal{A} \neq 0$.

Connection with $D=8$ F-theory / Heterotic string duality

What is the Heterotic elliptic curve?

$E_8 \oplus E_8$ standard fibration	$Spin(32)/\mathbb{Z}_2$ alternate fibration	an extra choice

In the context of period point in $\mathcal{H}_1$:
the Heterotic elliptic curves are just the factor elliptic curves E_1 and E_2 of the abelian surface $A = E_1 \times E_2$

What about the general N-polarized setting where $A = Jac(C)$?

In terms of modular invariants:

<u>SAME</u> ANSWER! $\{J(E_1), J(E_2)\}$

How do we prove these results?

Special Elliptic Fibration Structures with Section on N-polarized K3 Surfaces

X K3 surface

(ψ, S)

- $\psi : X \to \mathbb{P}^1$ proper map of analytic spaces
- S : choice of section of ψ

$\{$Isom classes of structures (ψ, S) on $X\}$

are in one-to-one correspondence with

$\left\{\begin{array}{c}\text{Primitive lattice embeddings } H \hookrightarrow NS(X) \\ \text{up to the action of Hodge isometries}\end{array}\right\}$

Lemma: Up to an isometry of N, there are exactly two non-isomorphic ways of primitively embedding H into N

Let $W :=$ orthog. compl. of image $W^{root} :=$ sublattice spanned by roots of W

Then the two cases are characterized by:

(a) $W = W^{root} \cong E_8 \oplus E_7$

(b) $W^{root} \cong D_{14} \oplus A_1$

$W / W^{root} \cong \mathbb{Z}/2\mathbb{Z}$.

So for an N-polarized K3 surface X there are two special types of elliptic fibrations with section $\Theta_1, \Theta_2 : X \to \mathbb{P}^1$ corresponding to $H \hookrightarrow N \xhookrightarrow{i} NS(X)$.

Θ_1, corresp to (a), is standard fibration

Θ_2, corresp to (b), is alternate fibration

Θ_1 : II^*, III^*

(for $\gamma \neq 0$)

Θ_2: I_{10}^*, I_2, two distinct sections

In terms of $Q(\alpha, \beta, \gamma)$, fibrations come from projections:

Θ_1: $[x, y, z, w] \mapsto [z, w]$

Θ_2: $[x, y, z, w] \mapsto [x, w]$

The canonical involution i_{SI} on $X(\alpha,\beta,\gamma)$ is induced by Θ_2-fiber-wise translation which exchanges the two sections. It is given in coordinates by the birational automorphism of the quartic surface

$$Q(\alpha,\beta,\gamma) \subset \mathbb{P}^3(x,y,z,w)$$

given by

$$[x,y,z,w] \mapsto [xz(w-2\gamma x), -yz(w-2\gamma x), w^3, zw(w-2\gamma x)]$$

The surface $Y(\alpha,\beta,\gamma) = \widetilde{X(\alpha,\beta,\gamma)/i_{SI}}$ also has a natural elliptic fibration structure with section, this time with an I_6^* fiber (there is no analogue of the standard fibration here any longer).

Geometric / computational task:

- Starting with smooth genus 2 curve C, construct Kummer surface $\mathrm{Km}(\mathrm{Jac}(C))$
- construct (the correct) I_5^* elliptic fibration with section on $\mathrm{Km}(\mathrm{Jac}(C))$
- construct an explicit isomorphism between $\mathrm{Km}(\mathrm{Jac}(C))$ (which depends on $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$) and $Y(\alpha, \beta, \gamma)$ by <u>matching</u> the elliptic fibrations with section
  - match the <u>positions</u> of corresp singular elliptic fibers
  - match the <u>J-invariants</u> of the corresp fiber elliptic curves

Use the realization of $\mathrm{Kum}(\mathrm{Jac}(C))$ via the $(16:6)$ configuration

E_i, $1 \le i \le 16$ exceptional curves

Θ_j, $1 \le j \le 16$ [illegible]

$\theta_m(Z, \cdot) = 0$

$m = (u, v)$

$u, v \in \{0, \frac{1}{2}\}^2$

$$\begin{tikzcd}
\mathrm{Jac}(C) \arrow[d, dashed] \arrow[dr, "\varphi_{|2C|}"] & \\
\mathrm{Kum}(\mathrm{Jac}(C)) \arrow[r] & \mathbb{P}^3
\end{tikzcd}$$

Fundamental theta functions $\theta_{m_i}(Z, \cdot)$

$m_1 = ((0,0),(0,0))$, $m_2 = ((0,0),(\frac{1}{2},\frac{1}{2}))$

$m_3 = ((0,0),(\frac{1}{2},0))$, $m_4 = ((0,0),(0,\frac{1}{2}))$

$\theta_{m_i}(Z, 2z)$ form a basis for

$i = 1,2,3,4$ $\qquad H^0(\mathrm{Jac}(C), \mathcal{O}_{\mathrm{Jac}(C)}(2C))$

Frobenius identities $\Rightarrow$ explicit quartic eqn

$$S_C \subset \mathbb{P}^3(x,y,z,w)$$

given as (Hudson quartic)

$$x^4+y^4+z^4+w^4 + 2D\,xyzw$$
$$+ A(x^2w^2+y^2z^2) + B(y^2w^2+x^2z^2)$$
$$+ C(x^2y^2+z^2w^2) = 0$$

where $A = \dfrac{b^4+c^4-a^4-d^4}{a^2d^2-b^2c^2}$, $B = \cdots$

$$D = \frac{abcd\,(d^2+a^2-b^2-c^2)\cdot\ldots\cdot(a^2+b^2+c^2+d^2)}{(a^2d^2-b^2c^2)(b^2d^2-c^2a^2)(c^2d^2-a^2b^2)}$$

with $a = \theta_{m_1}(Z,0)$, $b = \theta_{m_2}(Z,0)$,

$c = \theta_{m_3}(Z,0)$, $d = \theta_{m_4}(Z,0)$

Explicit form of the I_5^* fibration on $Km(Jac(C))$:

$$v^2 = u^3 - 2\,p(t)\,u^2 + (p(t)^2 - 4\,q(t))\,u$$

$$p(t) = t^3 - \frac{\mathcal{B}}{12}\,t + \frac{\mathcal{A}\mathcal{B} - 3\mathcal{C}}{108}$$

$$q(t) = -\frac{\mathcal{D}}{4}\left(t - \frac{\mathcal{A}}{24}\right)$$

Matching leads to formulas:

$$\gamma^6 = 2^{13} \cdot 3^5 \quad \frac{\mathcal{D}}{\mathcal{A}^5}$$

$$\alpha\gamma^2 = 2^4 \cdot \frac{\mathcal{B}}{\mathcal{A}^2}$$

$$\beta\gamma^3 = 2^6 \cdot \frac{(3\mathcal{C} - \mathcal{A}\mathcal{B})}{\mathcal{A}^3}$$

Applications and Extensions:

- Hodge and Tate Conjectures in explicit ex. for N-polarized K3 surfaces

- Construction of these explicit correspondences over number fields allows one to relate Galois actions on both sides

"Simplest" case:

CM abelian surfaces $\longleftrightarrow$ N-polarized K3 surfaces with $rk(N^{\perp}) = 2$?

(Elkies, Shioda, Schütt, Kumar)

- $E_1 \times E_2$ case $\Longrightarrow$ rk 20 K3 surface s.t. $NS^{\perp}$ defines binary quad. form
  E_1, E_2 CM of same type

Gauss product

Where next?

- "horizontally": other lattice polarized K3 surfaces which double cover Km(Jac(C))
- "vertically": other, lower rank, lattice polarizations

  (eg) Double cover of $\mathbb{P}^2$ branched on 6 lines

Abelian surface $\rightsquigarrow$ Kuga-Satake variety of X

- general case: use elliptic fibrations with section, and ideas from string duality

[Next year!]