Small Gaps Between Primes

Daniel Goldston

San Jose State University

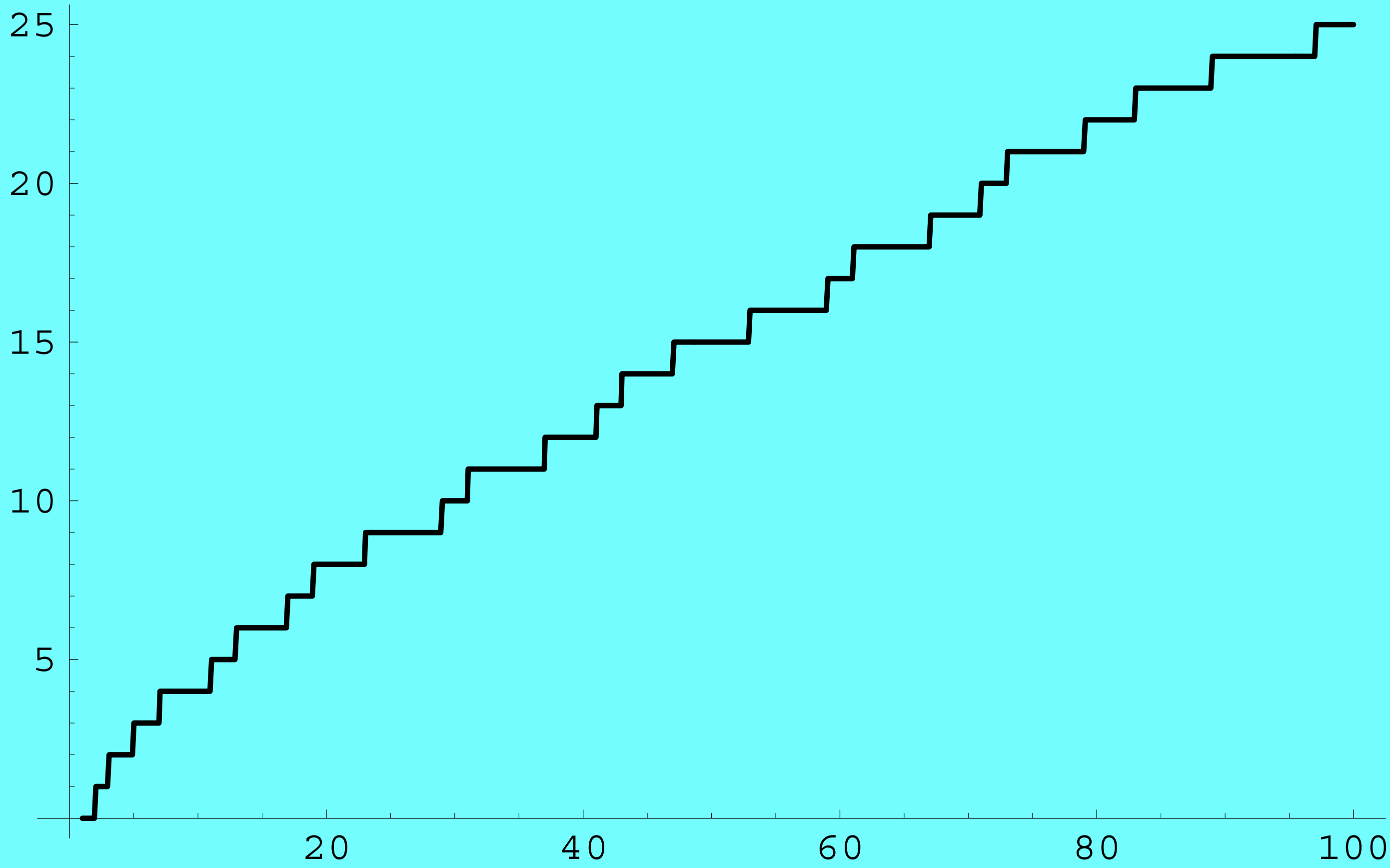
Joint work with:

Are there always primes much closer than the average ?

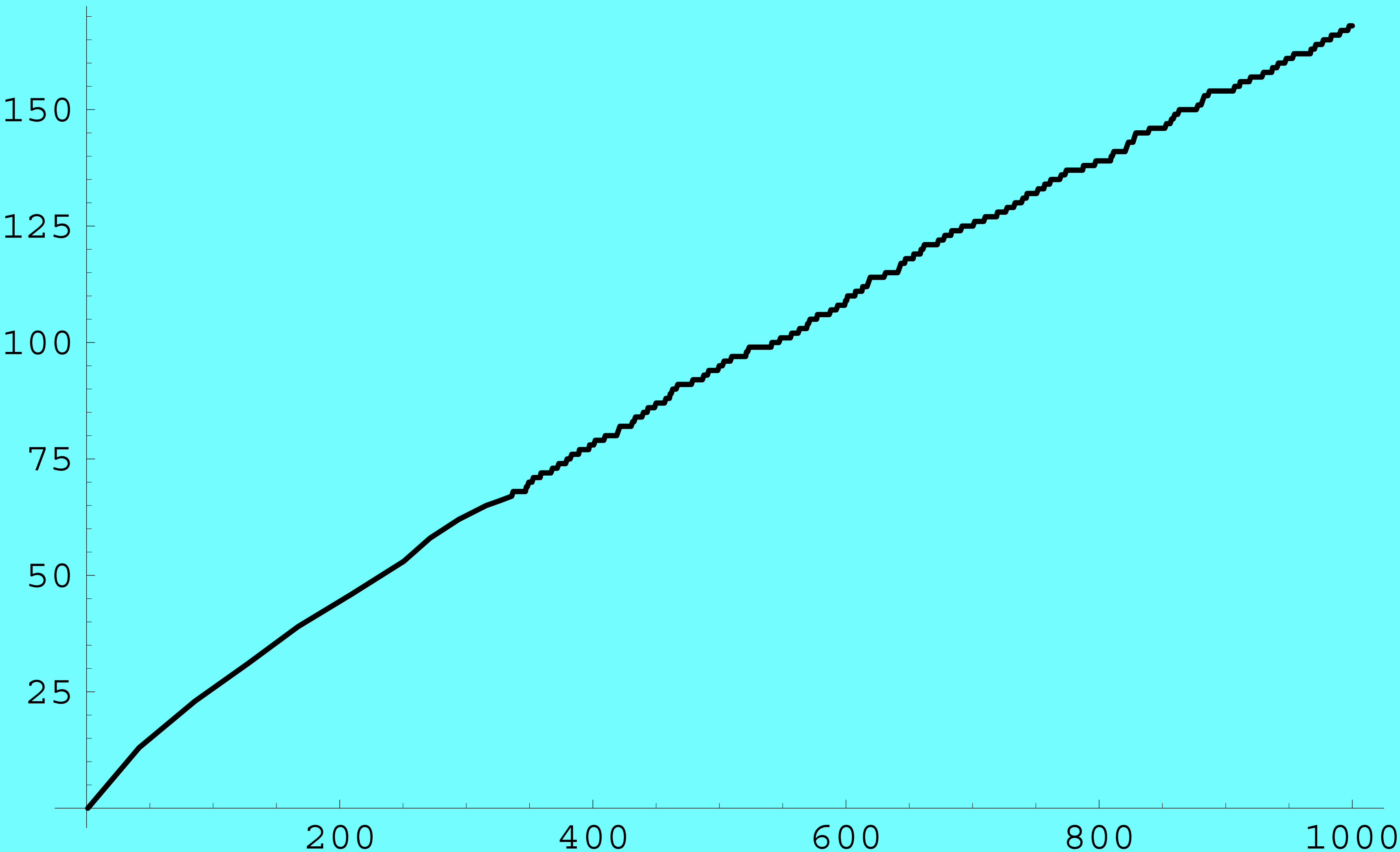
Let p_n denote the n-th prime.

The twin prime conjecture:

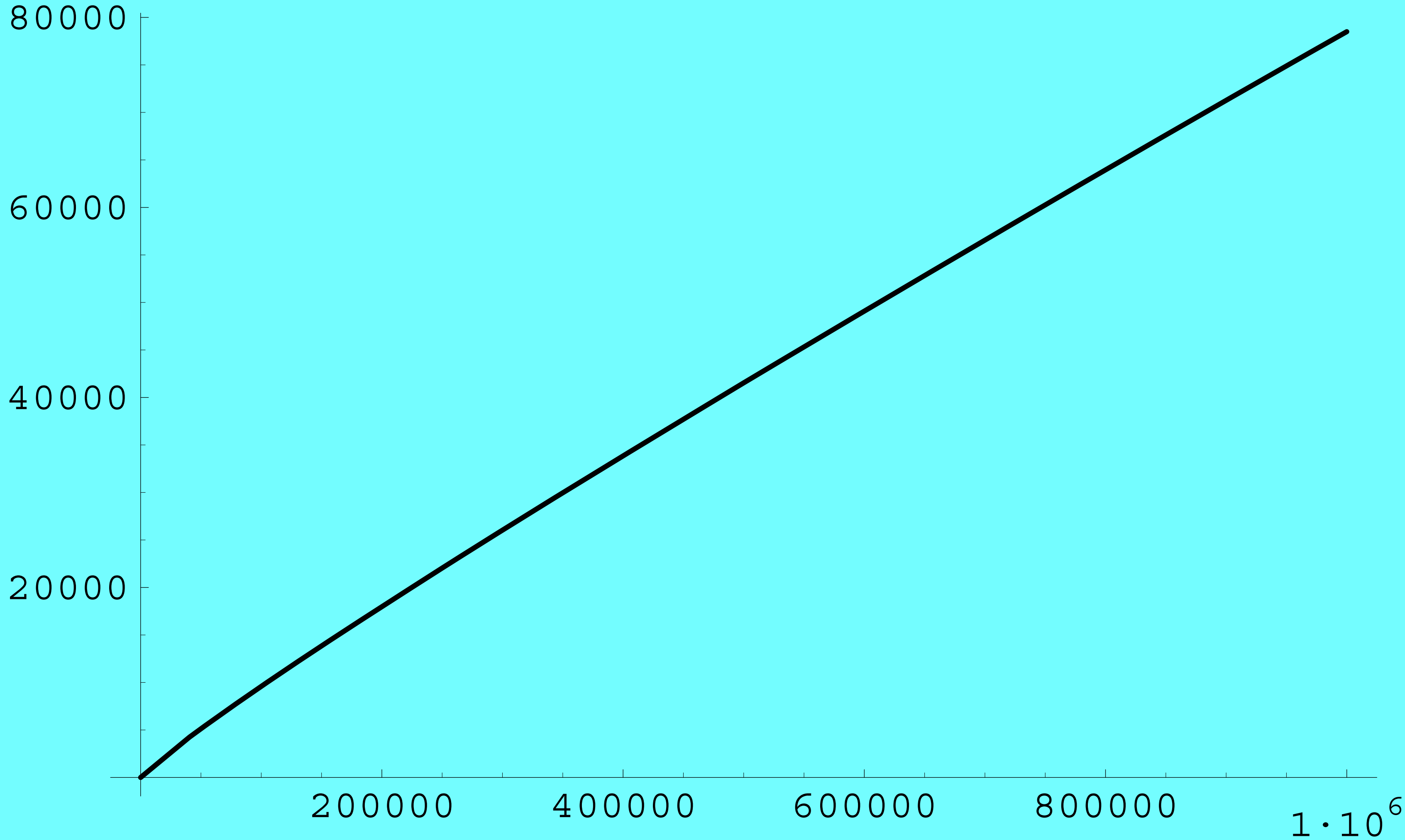
$p_{n+1} - p_n = 2$ infinitely often.



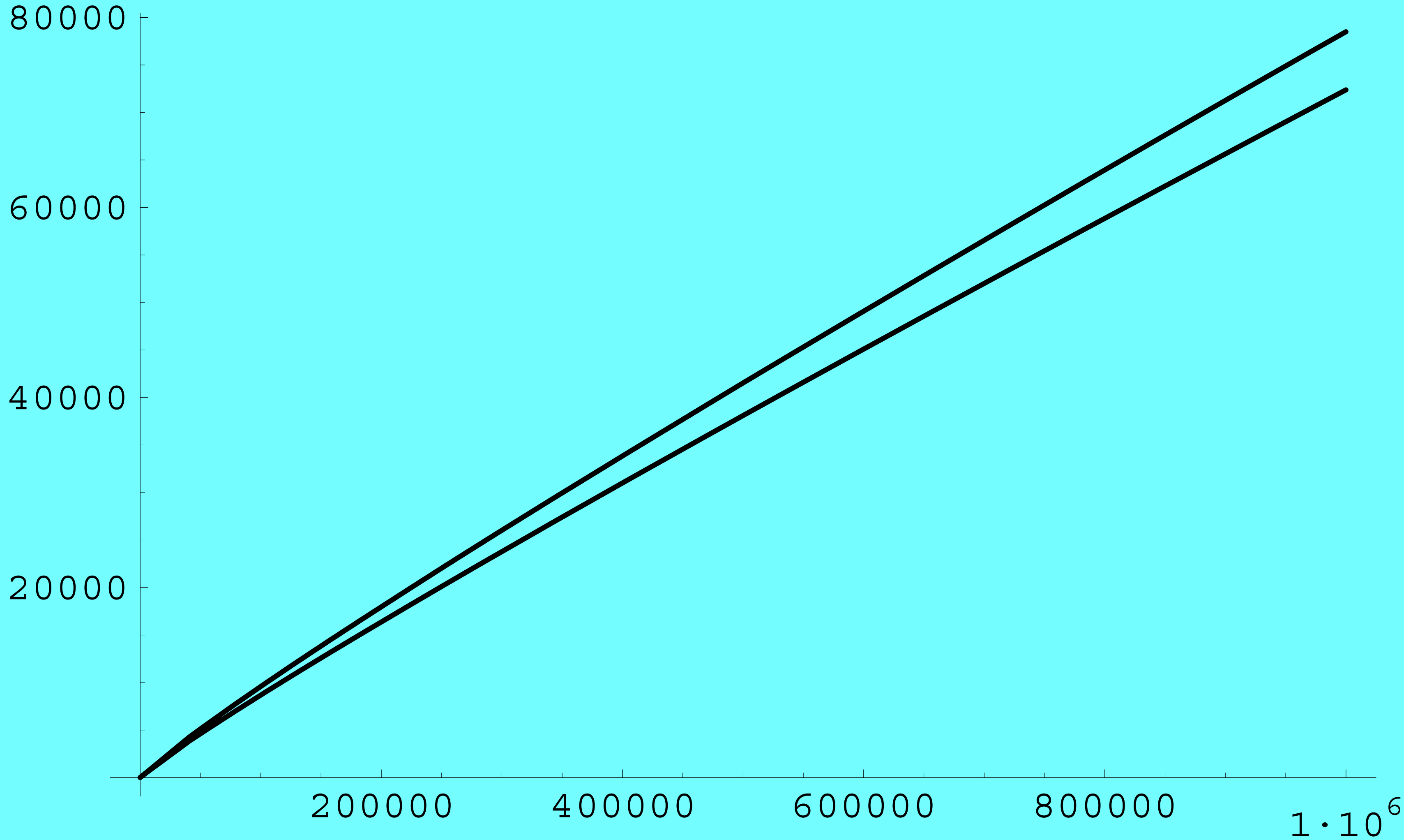
¼(x) = the number of primes · x.



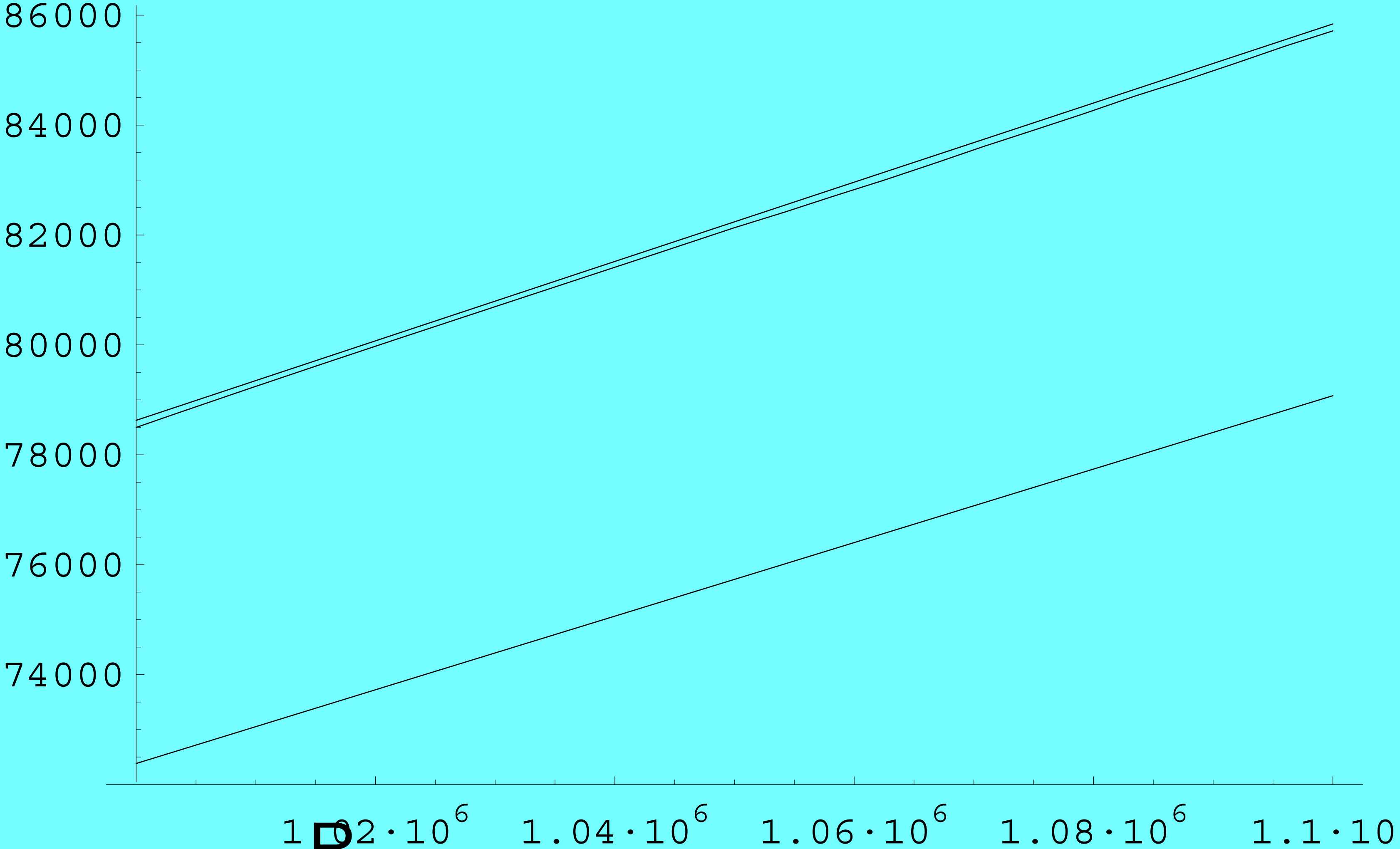
$\pi(x)$ = the number of primes $\le x$.



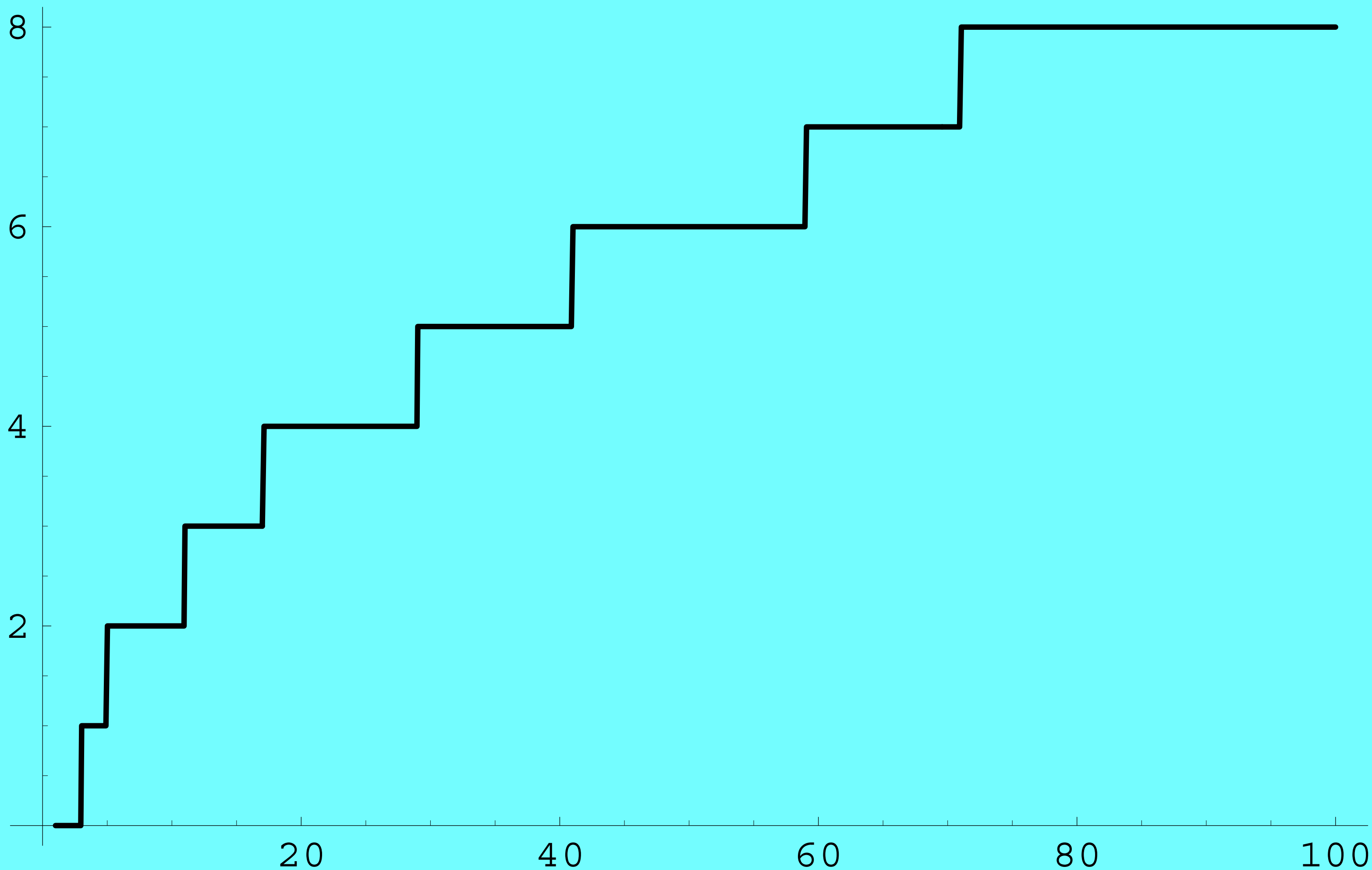
¼(x) = the number of primes · x.



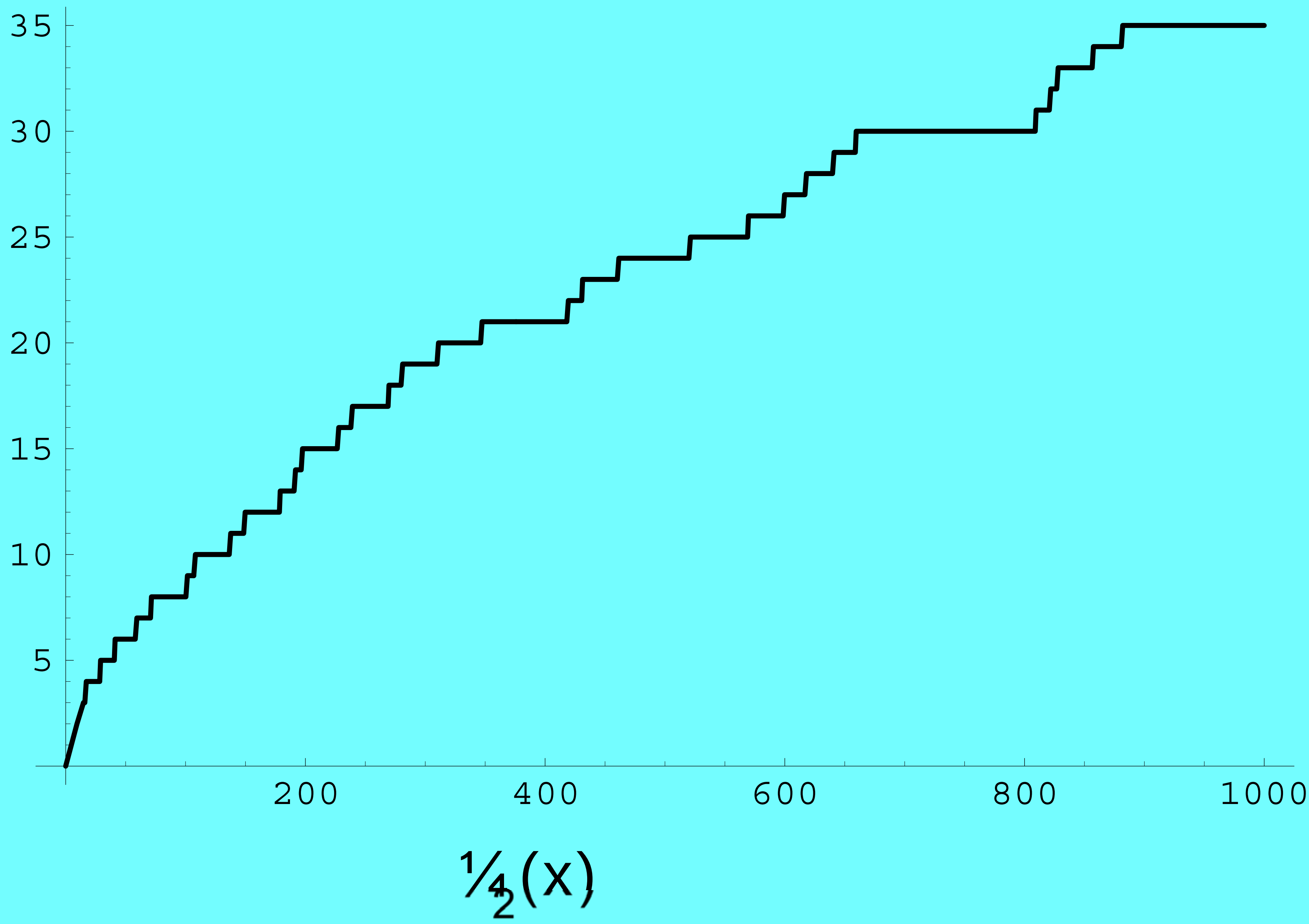
¼(x) is upper curve, $\frac{x}{\log x}$ is lower



Top: $\mathrm{li}(x) := \int_2^x \frac{1}{\log t}\,dt$, Middle: $\pi(x)$, Bottom: $\frac{x}{\log x}$



$\pi_2(x)$ = the number of twin primes $\le x$



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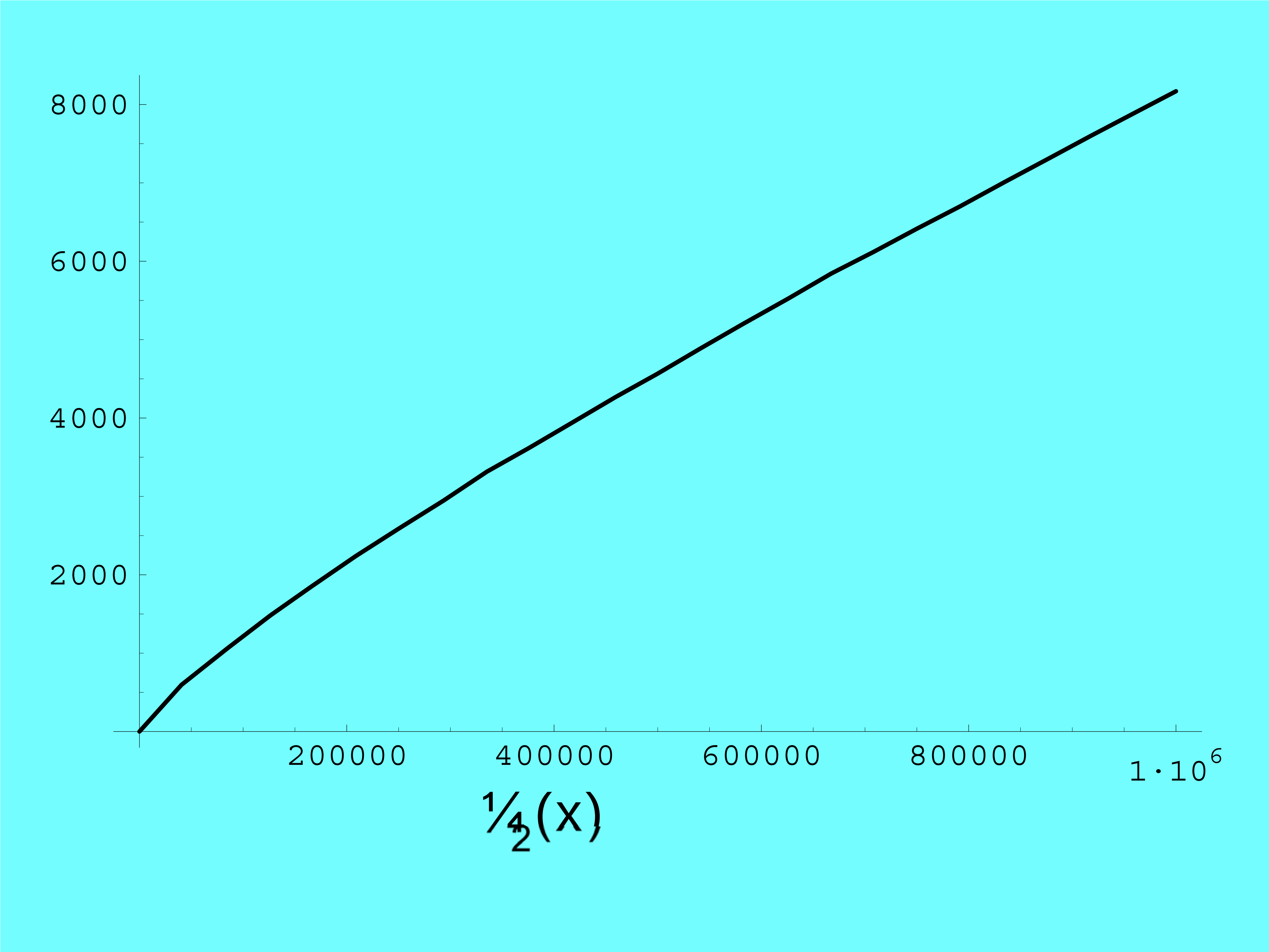
¼2(x)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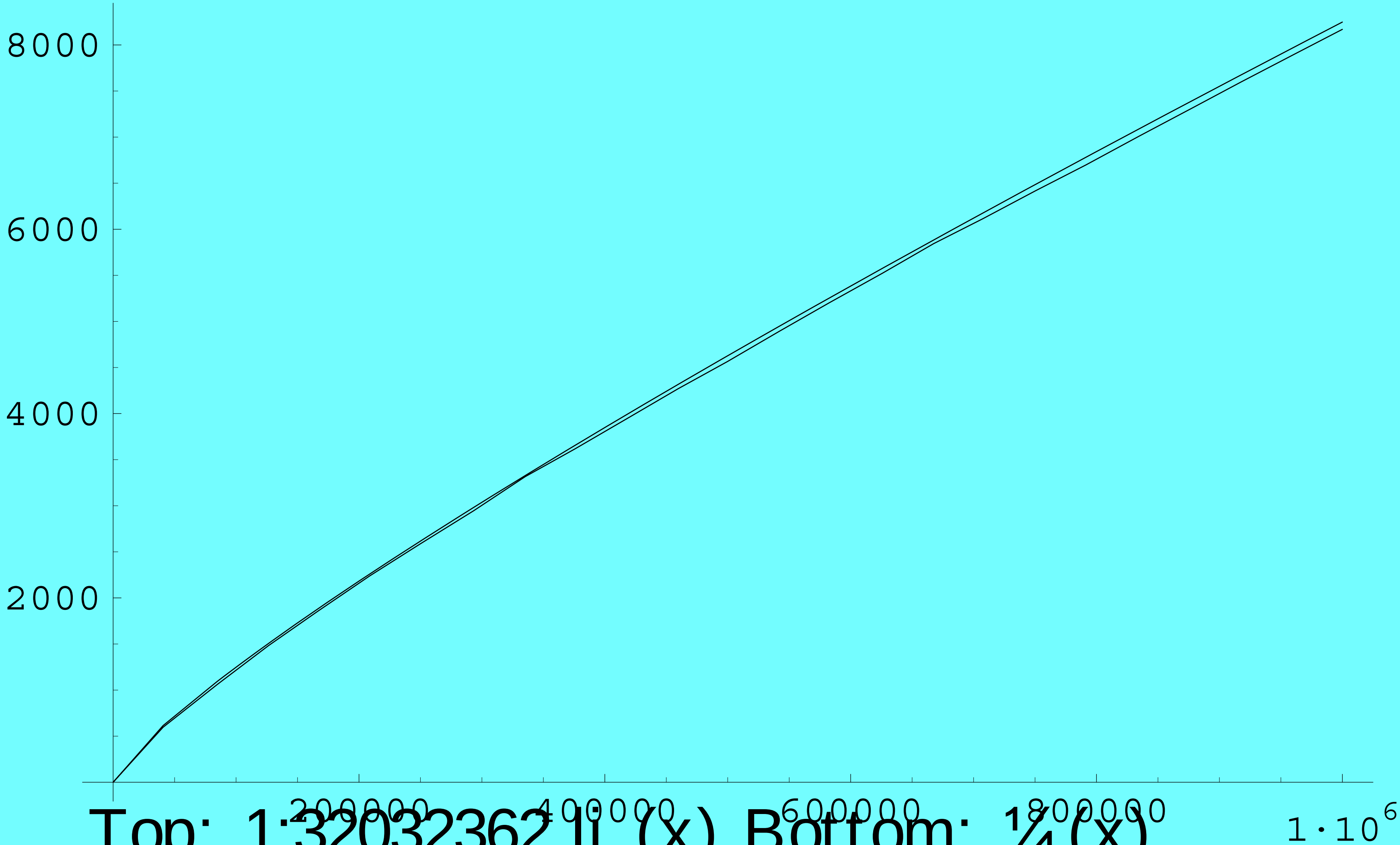

Top: $1.32032362\,\mathrm{li}_2(x)$ Bottom: $\pi_2(x)$

where $\mathrm{li}_2(x) := \int_2^x \frac{1}{(\log t)^2}\,dt$

More Modest Goal: Prove the existence of infinitely many consecutive primes whose distance apart is much smaller than the average distance: i. e. :

$$\Delta := \liminf_{n \to \infty} \left(\frac{p_{n+1} - p_n}{\log p_n} \right) = 0.$$

HISTORY

¢ · $\frac{2}{3}$; Hardy-Littlewood 1920's (on GRH)

¢ · $\frac{3}{5}$, Rankin 1940 (on GRH)

¢ · 1 ¡ c; ErdÄs 1940

¢ · $\frac{15}{16}$; Ricci 1954

¢ · $\frac{1}{2}$; Bombieri-Davenport 1965

¢ · 0:4665 : : : ; Bombieri-Davenport 1965

More History

- ¢ · 0:4571 : : : ; Pilt'ai 1972
- ¢ · 0:4542 : : : ; Uchiyama 1975
- ¢ · 0:4425 : : : ; Huxley 1975
- ¢ · 0:4393 : : : ; Huxley 1983
- ¢ · e_i ° = 0:5614 : : : ; Maier 1988
- ¢ · e_i ° (0:4425) = 0:2484 : : : ; Maier 1988
- ¢ · $\frac{1}{4}$; Goldston-Y³ld³r³m 2004

Goldston-Pintz-Yıldırım New Results

Theorem. (GPY)

$$\Delta := \liminf_{n \to \infty} \left(\frac{p_{n+1} - p_n}{\log p_n} \right) = 0.$$

Theorem. (Preliminary) For some $C > 0$,

$$p_{n+1} - p_n < C (\log p_n)^{1/2} (\log\log p_n)^2; \quad \infty \text{ often}.$$

Main limitation: We can not yet prove the same results for $p_{n+r} - p_n$ with $r \geq 2$.

G-P-Y Conditional Results

Our method uses information on primes in arithmetic progressions.

EXAMPLE. If you divide the natural numbers up modulo 3 you get three residue classes:

$$n \equiv 0 \pmod{3}: \quad 3, 6, 9, 12, 15, \ldots$$

$$n \equiv 1 \pmod{3}: \quad 1, 4, 7, 10, 13, \ldots$$

$$n \equiv 2 \pmod{3}: \quad 2, 5, 8, 11, 14, \ldots$$

We expect and can prove:

$$\pi(x;3,1) \sim \pi(x;3,2) \sim \frac{1}{2}\,\frac{x}{\log x} \quad \text{as } x \to \infty$$

The Prime Number Theorem for Arithmetic Progressions

$$\pi(x;q,a) \sim \frac{1}{\phi(q)}\,\mathrm{li}(x); \quad \text{as} \quad x \to \infty:$$

where $\phi(q) = |\{1 \le a \le q : (a,q) = 1\}|$;
is the Euler phi-function, and

$$\mathrm{li}(x) := \int_2^x \frac{1}{\log x}\,dx:$$

For applications we need $q = q(x) \to \infty$,
but unfortunately we only know this holds for
$q \le (\log x)^A$, for any A.

(On GRH $q \le x^{1=2}$ is acceptable.)

The Bombieri-Vinogradov Theorem

For any $\epsilon > 0$ and $A > 1$ we have

$$\sum_{q \le Q} \max_{\substack{a \\ (a,q)=1}} \left| \pi(x;q,a) - \frac{\mathrm{li}(x)}{\phi(q)} \right| \ll \frac{x}{(\log x)^A}$$

for $Q = x^{1/2}/(\log x)^{B(A)}$.

Level of Distribution of Primes in Progressions

If the above holds for $Q = x^{\theta - \epsilon}$ we say the primes have level of distribution θ.

BV implies $\theta = 1/2$ is true.

The Elliott-Halberstam conjecture: $\theta = 1$ is true.

Theorem. If the primes have level of distribution θ for a value of $\theta > 1/2$, then

$$p_{n+1} - p_n \le C(\theta) \quad \text{infinitely often.}$$

In particular, if $\theta \ge .971$

$$p_{n+1} - p_n \le 16 \quad \text{infinitely often.}$$

#	Gap
1	16
.95	20
.90	26
.85	32
.80	42
.75	60
.70	90
.65	158
.60	336
.55	1204

Prime Tuples

Let $H = \{h_1, h_2, \ldots, h_k\}$ be distinct integers, and consider the k-tuple $(n + h_1, n + h_2, \ldots, n + h_k)$.

If all the components of the tuple are primes, this is called a prime tuple.

H is called admissible if $h_1, h_2, \ldots, h_k$ never fill every residue class modulo p for all primes p.

Example: $H = \{0, 2\}$ is admissible; tuple $(n, n + 2)$, $H = \{0, 1\}$, $H = \{0, 2, 4\}$ are not admissible, with tuples $(n, n + 1)$, $(n, n + 2, n + 4)$.

Prime Tuple Conjecture: For admissible H, $(n + h_1, n + h_2, \ldots, n + h_k)$ is a prime tuple for infinitely many n.

Weak form of the Prime Tuple Conjecture

Theorem. If # > 1=2, then every admissible k-tuple with k ¸ c(#) contains at least two primes 1 often.

Limitation: The method only produces two primes in large enough admissible tuples, not three or more.

Hardy-Littlewood Prime Tuple Conjecture

Let $\nu_p(H)$ denote the number of distinct residue classes (mod p) the numbers $h \in H$ fall into.

Extend this to $\nu_d(H)$ for squarefree d by multiplicativity

Define the singular series

$$S(H) = \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\nu_p(H)}{p}\right)$$

H is admissible if and only if $\nu_p(H) < p$ for all p.

Let $\Lambda(n)$ denote the von Mangoldt function, defined to be $\log p$ if $n = p^m$ and zero otherwise

Define the prime tuple counting function

$$\Lambda(n; H) = \Lambda(n + h_1)\Lambda(n + h_2)\cdots\Lambda(n + h_k).$$

Hardy and Littlewood conjectured that for H admissible,

$$\sum_{n \le N} \Lambda(n; H) = N\left(S(H) + o(1)\right), \quad \text{as } N \to \infty.$$

Approximating Prime Tuples: The Old Way

The simplest approximation of $\Lambda(n)$ is based on the elementary formula

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d}$$

This sum has too many terms to be useful, so we approximate with the smoothly truncated divisor sum

$$\Lambda_R(n) = \sum_{\substack{d|n \\ d \le R}} \mu(d) \log \frac{R}{d}.$$

Then our approximation for $\Lambda(n;H)$ is

$$\Lambda_R(n;H) = \Lambda_R(n+h_1)\Lambda_R(n+h_2)\cdots\Lambda_R(n+h_k).$$

We next work out formulas for the sums

$$\sum_{n \le N} \Lambda_R(n;H)^2; \quad \text{and} \quad \sum_{n \le N} \Lambda(n+h_0)\Lambda_R(n;H)^2.$$

The main terms are complicated but standard. The error term is $O(R^{2k}(\log R)^C)$ for the first. We need this to be $O(N)$ forcing $R = N^{1/2k-\epsilon}$ which is very short. For the second sum we need $R = N^{\theta/2k-\epsilon}$ which is even shorter.

Results from using Old Approximation

Optimizing gives $c \le \frac{1}{4}$

Green and Tao used the asymptotic formula for

$$\sum_{n \le N} \Lambda_R(n;H)^2$$

in their work on arithmetic progressions of primes.

Ironically, the new approximation does not work for Green-Tao.

Approximating Prime Tuples: The New Way

In 2003 we tried to use an approximation directly on tuples, but it wasn't smooth enough.

The right idea came out of a paper of Heath-Brown (1997) and is standard in sieve theory

Instead of the tuple $(n + h_1; n + h_2; \ldots; n + h_k)$ consider the polynomial

$$P(n; H) = (n + h_1)(n + h_2) \ldots (n + h_k)$$

Then the tuple $(n + h_1, n + h_2, \ldots, n + h_k)$ is a prime tuple when the polynomial

$$P(n, H) = (n + h_1)(n + h_2)\ldots(n + h_k)$$

has k prime factors.

The generalized von Mangoldt function

$$\Lambda_k(n) = \sum_{d \mid n} \mu(d)\left(\log \frac{n}{d}\right)^k$$

is non-zero on P_k's, but vanishes otherwise, so

$$\Lambda_k(P(n, H)) \quad \text{detects prime tuples}$$

New Approximation

$$\Lambda_R(n;H) = \frac{1}{k!} \sum_{\substack{d|P_H(n) \\ d \le R}} \mu(d) \left(\log \frac{R}{d} \right)^k .$$

By chance this is extremely close to the failed approximation of 2003.

However, this approximation only giving $\Delta \le .135\ldots$.

What makes everything work

To detect SOME primes in tuples, you only need to show that $P(n;H)$ has $\le k+\ell$ prime factors, for some $\ell < k$. Thus we should approximate $\Lambda_{k+\ell}$.

Hence define

$$\Lambda_R(n;H;\ell) = \frac{1}{(k+\ell)!} \sum_{\substack{d \mid P(n;H) \\ d \le R}} \mu(d) \left(\log \frac{R}{d}\right)^{k+\ell}.$$

Theorem 1. For $R \leq N^{1/2}/(\log N)^{2k}$ and $h \leq R^{\epsilon}$ with $R, N \to \infty$ we have

$$\sum_{n \leq N} \Lambda_R(n; H; \ell)^2 \sim \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k+2\ell)!} \mathfrak{S}(H) N.$$

Theorem 2: If primes have level of distribution θ and $R \le N^{\theta/2-\epsilon}$, then for $h \le R^{\epsilon}$ with $R, N \to \infty$

i) If $h_0 \notin H$;

$$\sum_{n \le N} \lambda_R(n;H;\ell)^2 \theta(n+h_0) \sim \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k+2\ell)!} S(H \cup \{h_0\}) N;$$

ii) If $h_0 \in H$ (get $\ell \to \ell+1$ and $k \to k-1$)

$$\sum_{n \le N} \lambda_R(n;H;\ell)^2 \theta(n+h_0) \sim \binom{2\ell+2}{\ell+1} \frac{(\log R)^{k+2\ell+1}}{(k+2\ell+1)!} S(H) N.$$

Sketch of Proof of Small Gaps

For $\ell \geq 0$ and $R = N^{\vartheta/2-\varepsilon/4}$, we have

$$S := \sum_{n=N+1}^{2N} \left(\sum_{i=1}^{k} \Lambda(n+h_i) - \log 3N \right) \Lambda_R(n;H;\ell)^2$$

$$\sim k \binom{2\ell+2}{\ell+1} \frac{(\log R)^{k+2\ell+1}}{(k+2\ell+1)!} S(H)N$$

$$- \log 3N \binom{2\ell}{\ell} \frac{(\log R)^{k+2\ell}}{(k+2\ell)!} S(H)N$$

$$S \geq \left(\frac{2k}{k+2\ell+1} \frac{2\ell+1}{\ell+1} \log R - \log 3N \right) M$$

$$\geq \left(\frac{k}{k+2\ell+1} \frac{\ell+1/2}{\ell+1} \cdot 2\theta - 1 - \epsilon \right) M \log 3N,$$

where $M = \frac{(\log R)^{k+2\ell}}{(k+2\ell)!} \binom{2\ell}{\ell} \mathfrak{S}(H) N.$

Thus we need $\frac{k}{k+2\ell+1} \frac{\ell+1/2}{\ell+1} \cdot 2\theta > 1$.

Clearly we can satisfy this for large k and $\ell = o(k)$ for any $\theta > \frac{1}{2}$.

In particular, $S > 0$ if $\ell = 1$ and $k = 7$ when $\theta > \frac{20}{21}$.

$\Delta = 0$ unconditionally

We just fail to prove the result unconditionally with $\theta = 1/2$, so how can we “win an ε”? Consider

$$\sum_{n=N+1}^{2N}\left(\sum_{1\le h_i\le h}\Lambda(n+h_i)-\log 3N\right)\times\sum_{\substack{1\le h_1,h_2,\ldots,h_k\le h\\ \text{distinct}}}\Lambda_R(n;\mathcal{H};\ell)^2$$

A result of Gallagher for the average of the singular series gives an addition factor of h. Hence with $\vartheta = 1/2$ this is positive if $h > \varepsilon \log N$.

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USA TODAY
Life
SECTION D
Tuesday, March 25, 2003

All that buzz from the Oscar bashes, 2D

Winning conversation: Adrien Brody and Nicole Kidman at *Vanity Fair's* 2003 Oscar party.

By Eric Charbonneau/BEImages

Prime-number analysis sees advances

Mathematicians report the biggest advance in analyzing gaps between prime numbers in four decades. Prime numbers, ones like 3 or 7 or 1087, have fascinated number theorists from the ancient Greeks to codemakers today because they can only be divided by one and themselves. Calculating the frequency with which they occur from 1 to infinity is perhaps the most famous application of number theory. Dan Goldston of San Jose State University and Turkish mathematician Cem Yildirim will report this week at an American Institute of Mathematics meeting that they have developed a formula that limits the size of gaps between primes with unparalleled precision. They hope the work leads to new ways to find so-called "twin" primes, such as 11 and 13, that are separated by only one number. Mathematicians hope unveiling the secrets of prime numbers will lead to a fundamental understanding of all types of math.

From staff and wire reports
E-mail: Betterlife@usatoday.com

SOLUTION | SJSU professor solves problem in prime number theory

Continued from Page 1B

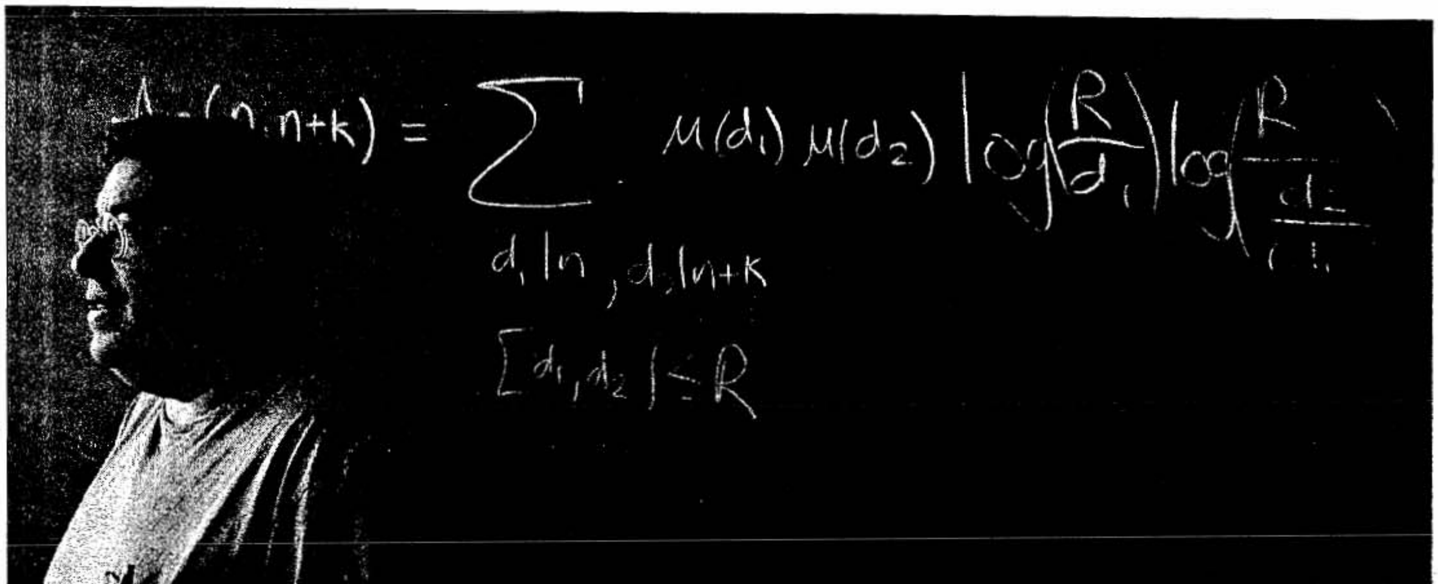
"That's about a 17,000-digit number," Goldston said. "They're not hard to find." At any given moment, in fact, the top 20 twin primes can be found on the Web at **www.utm.edu/research/primes/lists/top20/twin.html**.

The question is, do these twin primes keep occurring indefinitely, up into the realm of zillions and bazillions? Or do they just fizzle out at some point?

Most mathematicians suspect they keep going, off into infinity. But proving that has been infinitely difficult. So Goldston picked off a more manageable piece of the problem: Can you always find prime numbers that may not be twins, but that are much closer together than average? Taking into account, of course, the fact that the bigger numbers get, the sparser primes become.

Working with Cem Yalcin Yildirim of Bogazici University in Istanbul, Goldston was finally able to say yes.

"It's hard to know where this is going to lead," Montgomery said. "It could be that this is going to open up



RICHARD KOCI HERNANDEZ — MERCURY NEWS

Professor Dan Goldston displays his solution to a problem in prime number theory Tuesday on a blackboard at the San Jose State campus. Mathematicians described the advance as the most important breakthrough in the field in decades.

Conrey said. "A lot of the excitement is, we don't know how far this thing is going to go. There are going to be a lot of applications, I think."

Goldston seems a bit surprised by it all. Colleagues describe him as funny, in a low-key kind of way — not the kind of guy who pushes himself forward. But they add that he has been amazingly prolific, writing 33 mathematical papers over the years despite a full-time teaching load.

A hint of his sense of humor can be found on his Web site, which features a photo of Goldston, seemingly dozing off, as two small kids climb on his back. He and his wife, Ryoko, have three children — Shota, 7, Aiko, 5, and Makoto, 3.

The couple met when she took one of his classes at San Jose State.

"He's so funny. He's always joking around. His lectures are hilarious," Ryoko said.

"He's sort of working on this while we watch TV," she said. "He doesn't even have a study. His study is the dining room table. I think he's always thinking about math in his head, but he doesn't show it too much.

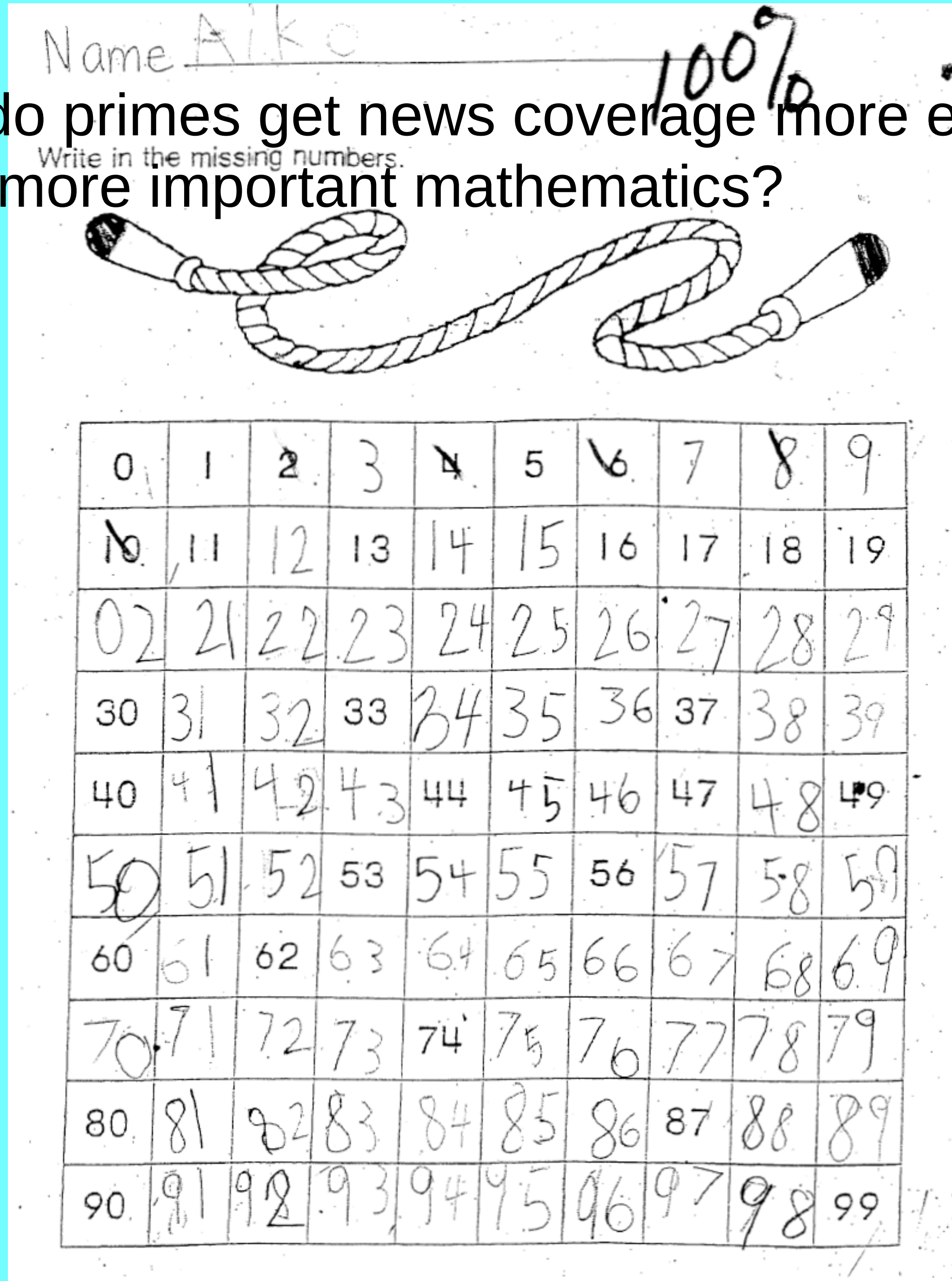
"He doesn't even own a suit. Maybe after this we're going to have to get him a suit or two."

IF YOU'RE INTERESTED

Dan Goldston's Web site is at www.math.sjsu.edu/~goldston. An account of his work is at [illegible] math.org/

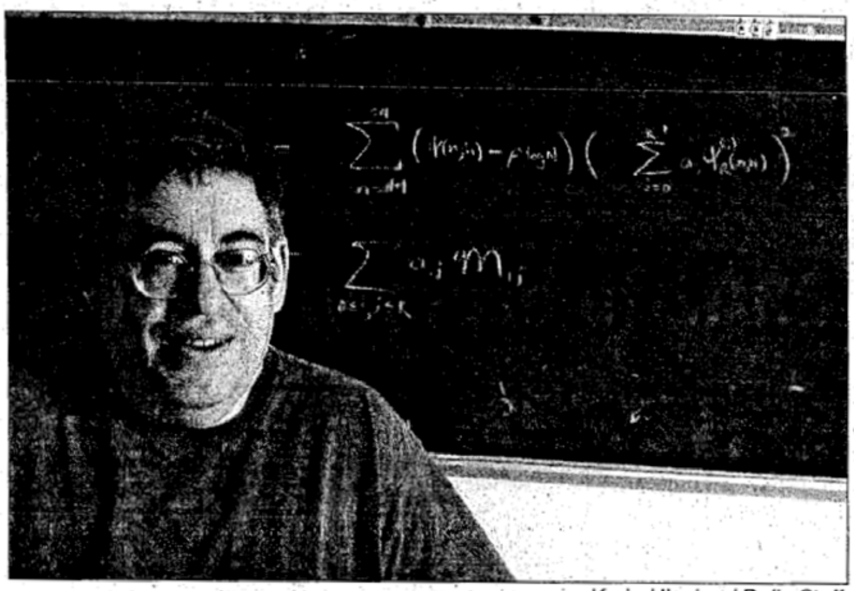
Contact Glennda Chui at gchui@mercurynews.com or (408) 920-5453.

Why do primes get news coverage more easily than other more important mathematics?



Math professor solves equation

By Norikazu Ambo
Daily Staff Writer



Karin Higgins / Daily Staff

Math Professor Dan Goldston solved a controversial prime numbers equation that he had been working on for 20 years.

Repeating mistakes and making a little bit of progress in two decades of his struggle, San Jose State University veteran math professor Dan Goldston solved one of the most controversial problems in the prime number theory last month with support from his Turkish partner.

"I knew I had a right idea," said Goldston, who penetrated the patterns of prime twins or prime pairs — numbers that differ by 2 such as a pair of 3 and 5 and 11 and 13 — and discovered an approximation, which was officially announced at a conference in Oberwolfach, Germany on March 13.

It took a total of 20 years to unearth the approximation for him and his fellow mathematician, Cem Yalcin Yildirim at Bogazici University in Istanbul.

"You just have to keep working," he said.

According to Goldston, the topic of the primes had been one of the well-recognized mathematical enigmas since the era of Euclid's Elements more than 2,000 years ago.

He said the toughest part in his research was to "come up with an argument and demonstrate the truth."

"It's easy to observe the trend of numbers, but to find a new property, we observe, is different," he said.

The prime numbers are the whole numbers not divisible by any positive integer other than itself and one without a remainder, such as 2, 3, 5, 7, 11, 13 and so on.

Brian Conrey, an executive director of the American Institute of Mathematics, who Goldston told about his innovative breakthrough before his formal announcement at the assembly, said that Goldston's work "opened

He said his enjoyment in math comes from plowing through problems by investigating numbers, geometry, formulas and rules.

"What's fun about math is to answer difficult questions and prove certainty of them," he said.

Holding his research paper, titled "Small Gaps Between Primes," co-authored by Yildirim, Goldston exuded passion for his profession, asking "Do you want to write a 68-page paper about the thing you don't like?"

When he experienced a few minutes of realization last November about his rough estimate, he immediately told his wife Ryoko that he "just had an amazing idea."

That idea triggered the solution, he said.

Ryoko sensed his excitement, saying he was very excited.

"Dan is usually a laid-back and

with applause.

She said Goldston reacted shyly, saying, "You guys can stop that?"

Yuan said she enjoys his class because his teaching keeps students alert.

"There is no need to take a roll in his class because everyone wants to come and pay attention," she said. "Basically, his class is not boring."

Students Tony Chiang, a sophomore electrical engineering major, Tyler Zeng and Jennifer Byun, both sophomore computer science majors, agreed that Goldston has a sense of humor and keeps students entertained.

For example, Goldston once created his own terminologies on integrators, referring to them as "front slap" and "back slap," Byun said.

"He is a funny teacher," Zeng said.

Chiang said, "He pretends to be mean in the class, but we know he is not."

For Byun, paying attention in his class

Walking for can

Jennifer Fiamingo and Darin Smith, members o relay supported by the American Cancer Socie 10 a.m. Saturday to 10 a.m. Sunday.

QUESTIONS| *Ch*

continued from page 1

"We have 60 departments that go by the contract," Hill said. "You have to have two classes evaluated per academic year, and, if you want to do

returned it along with a mate to deliv Brent said.

"The Acade in March for

SCIENCE NEWS

(Calif.) State University and Cem Y. Yildirim of Bogaziçi University in Istanbul have proven something much stronger: Given any fraction, no matter how small, there are infinitely many prime pairs closer

It's long been suspected that human sperm sense chemicals secreted by an egg. More than a decade ago, for example, investigators found that human sperm sport proteins called olfactory receptors, the same

This Week

Prime Finding

Mathematicians mind the gap

A mathematical duo has made a surprising advance in understanding the distribution of prime numbers, those whole numbers divisible only by themselves and 1. The new result is the most exciting work on prime numbers in more than 3 decades, says mathematician Hugh L. Montgomery of the University of Michigan in Ann Arbor. However, he cautions that experts are still checking the details of the proof.

Among small numbers, primes are common. Of the first 10 numbers, for instance, 4 of them—2, 3, 5, and 7—are prime. But

Pomerance of Bell Laboratories in Murray Hill, N.J. "It's an end-run around the big plan for development in the field."

Brian Conrey, director of the American Institute of Mathematics in Palo Alto, Calif., agrees. "It's an incredible breakthrough," he says.

Goldston and Yildirim's novel idea was to examine the distribution not just of pairs of primes, but also of triples, quadruples, and larger groupings. Studying this wider question simplified the formulas estimating the spacing of primes, and to the team's surprise, the new result about smaller-than-average prime gaps fell out.

"The result was so much better than what we expected, I almost thought we had made a mistake," says Goldston

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cess to prove the conjecture.

However, mathematicians have had some success in considering the more general case of primes that are closer together than predicted by the average-spacing formula. In 1965, Enrico Bombieri of the Institute for Advanced Study in Princeton, N.J., and the late Harold Davenport proved there are infinitely many pairs of primes that are

D. SPEHR/SCIENCE

By a Nose?

Human sperm may sniff out the path to an egg

A man's sperm seek out a variety of floral scents, suggesting that these microscopic swimmers possess a primitive kind of nose

Since the scientists don't think that eggs make bourgeonal, they're now searching the fluid from women's reproductive tract for a natural stimulus for the new sperm receptor. "The natural sperm attractant might be structurally related to bourgeonal," says Spehr. "One can only speculate if it is secreted by the egg itself or the cumulus cells surrounding the egg or even by cells workers recreated th

NUMBER THEO

Prime Pr

PALO ALTO, CALIFORN breakthrough, an A mathematician have havior of enormou dramatically sharp speaking at the Ame ematics (AIM) here 50 number theorists about the new resul

Science

SPECIAL ISSUE

BIOLOGICAL IMAGING

Light microscopy reveals the complexity and dynamic nature of cell division. The proper segregation of the replicated chromosomes (blue) requires that they all first become aligned near the equator of the mitotic spindle (yellow). [Image: C. L. Rieder and A. Khodjakov]

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VIEWPOINT

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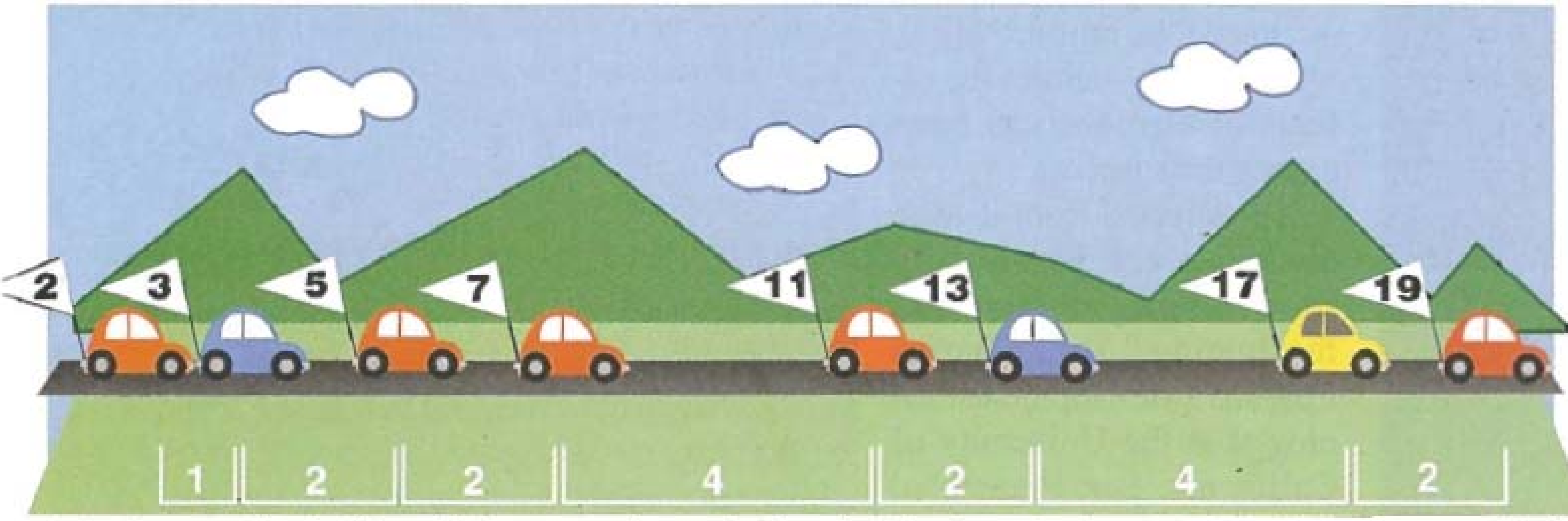

Fast track. Studies of clumps and gaps in the distribution of prime numbers have been stalled for decades, but a new approach may give the field a jump-start.

The distributions integers that can be themselves and 1—h for centuries. Primes such as the numbers 113, or at huge interv ry's most celebrated c Prime Conjecture, sta those that crop up tw

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However, We have thinking ey, direc- have the the first we have

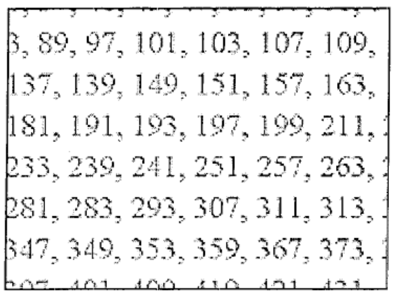
ACKENZIE California.

CREDIT: J. MOGLIA/SCIENCE

Friday, 4 April, 2003, 11:11 GMT 12:11 UK

Prime number breakthrough

By Dr David Whitehouse
BBC News Online science editor



A pair of mathematicians has made a breakthrough in understanding so-called prime numbers, numbers that can only be divided by themselves and one.

Other mathematicians have described the advance as the most important in the field in decades.

It was made by Dan Goldston, of San Jose State University, and Cem Yildirim, of Bogazici University in Istanbul, Turkey. It has just been announced at a conference in Germany on Algorithmic Number Theory.

The advance is related to an idea called the twin prime conjecture. This idea, still unproved, is that there are an infinite number of pairs of prime numbers that differ only by two.

Number building

"Neither of us ever expected to get particularly good results by this method. It's actually completely amazing to me," says Goldston.

Commenting on the breakthrough, Hugh Montgomery, a mathematician at the University of Michigan in Ann Arbor, US, says that Goldston has really broken a barrier.

Dear Dan,

I really enjoye

what a wond

teaching, rese

so pleased tha

Best wishes f

Cordially,

Bob Caret

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SAN FRANCISCO

March 28, 2003

Professor Daniel A. Goldston
Department of Mathematics
San Jose State University
One Washington Square
San Jose, CA 95192

Dear Professor Goldston:

Congratulations on your recently announced breakthrough in prime number theory. The work accomplished by you and your Turkish colleague after so many years of effort is truly outstanding and underlines the high quality of California State University faculty. I am proud to join your campus community—the faculty, staff, students, and alumni—in offering you both heartfelt applause and best wishes for continuing success in your field.

With kind regards,

Sincerely,

Charles B. Reed
Chancellor

CBR:da

cc: President Robert L. Caret

March 27, 2003

rful story, and

niversity life –

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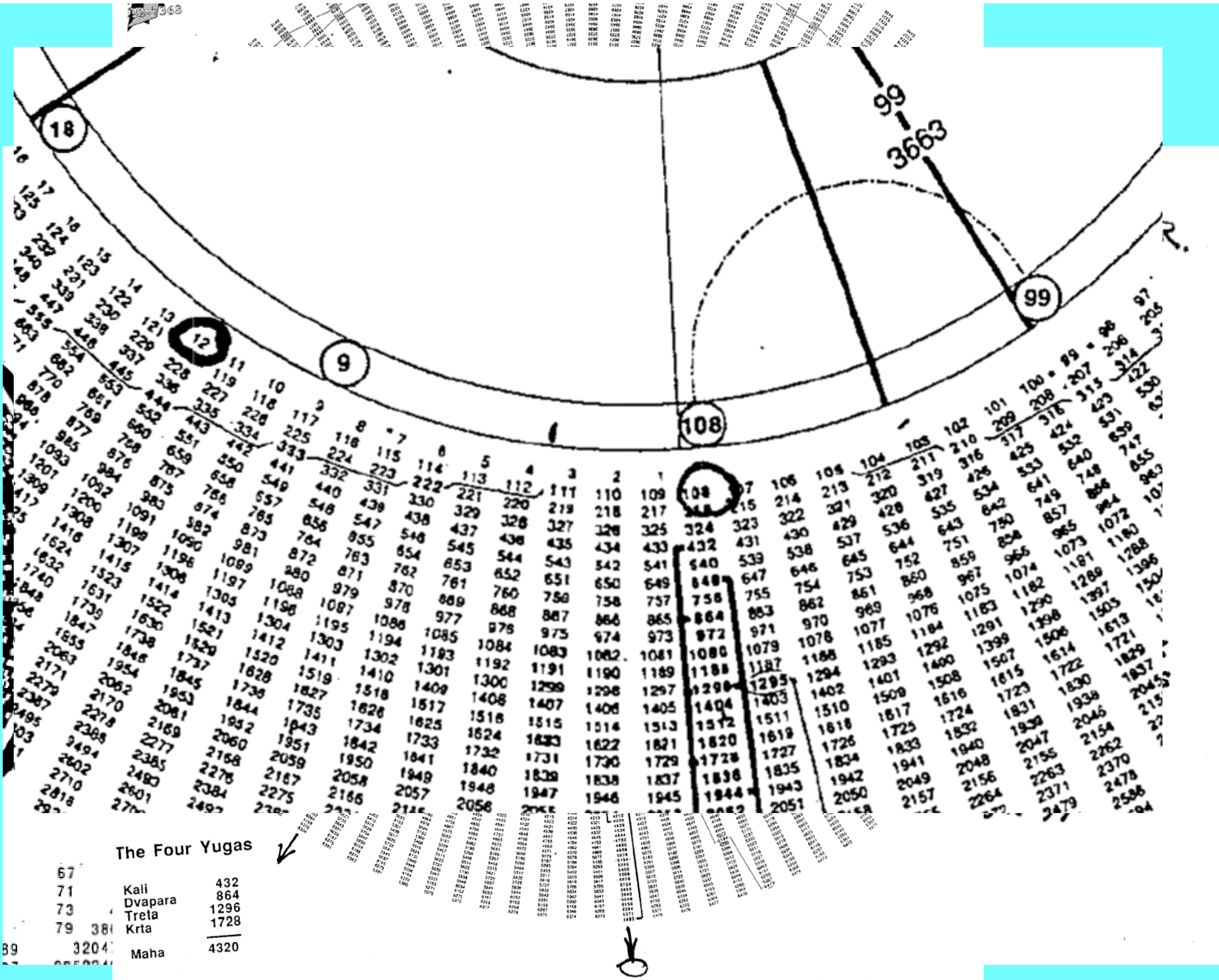


The Four Yugas

Kali	432
Dvapara	864
Treta	1296
Krta	1728
Maha	4320

The Year in Science

2003 YEAR IN SCIENCE

2003: Mathematicians Face Uncertainty

8

MATHEMATICS – This will surely be remembered as the year mathematicians finally had to agree that their prized notion of "absolute proof" is an unattainable ideal--an excellent goal to strive for, but achievable only in relatively simple cases. Moreover, they were forced to make this adjustment under the harsh glare of the media, following three major news stories about so-called mathematical proofs.

Early in the year, American mathematician Daniel Goldston and his Turkish colleague Cem Yildirim announced a proof of the twin prime conjecture, which says there are an infinite number of prime numbers differing by two, such as 3 and 5, or 11 and 13. Although experts around the world initially agreed that the new proof was correct, a few weeks later an insur-

in the proof, they were still not sure that it was correct. The journal agreed to publish Hales's proof, but only with a disclaimer saying they were not sure that it was right.

What all three episodes reflect is the complexity and abstraction of many modern proofs. Even the experts find

is not clear that anyone has ever seen such a thing. The examples familiar to most people from their high school math classes are the geometric arguments Euclid presented in *Elements*. But as the German mathematician David Hilbert pointed out in the late 19th century, many of those arguments are logically incorrect. Euclid made repeated use of axioms that he had not stated, without which his arguments are not logically valid.

It took some effort—following a delay of more than 2,000 years—for Hilbert to present correct proofs. Today most mathematicians regard his arguments as valid right-wing proofs, myself included. But if you push me to say how I know this, I will end up mumbling that his arguments convince me and have convinced all the

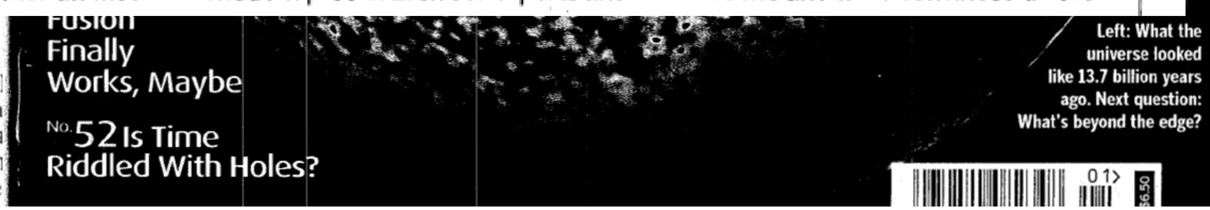
MEDICINE – In January a veter narian noticed a drawn and sickl 8-year-old cow at a slaughterhous in Alberta. Tests later showed th the animal had been suffering fro North America's first homegrown cas of bovine spongiform encephalopath or mad cow disease.

Canadian minister of agricultur Lyle Vanclief insists that the Albert cow's remains never made it into th food chain. But the United State which until then consumed 78 percer of Canada's beef exports, banne Canadian beef. And an internation panel of veterinary scientists an infectious-diseases specialists rec ommended that additional safeguard be put in place in North America: re moval of brain and spinal-cord tissu

Mad Cow Disease

hidden error.

In March, Daniel Goldston San Jose State University and associate Cem Yildirim of Boga i University in Istanbul ma headlines by announcing th d overcome a major obstacle



discoveries. For example, Einstein's relativity was the abstract geometry of century," Granville said.

' unyielding support

ton and Yildirim's col-

selves open ing band o who suspe sciences, rather th truth. To neat, logic history of man's best friend.

The findings should aid in tracking down disease genes, says Ostrander. She can now expand the search for a gene in one breed to other breeds shown to be related by their microsatellite compositions. Having a larger sample will make it easier to detect the mutation at fault. "This is what I see as the most powerful use of the data," she notes.

The dog offers other advantages over humans for gene hunts, says Sutter. To find the mutated genes underlying complex diseases such as cancer, geneticists look for base changes along the DNA where the implicated gene seems to be. Initial analyses suggest that geneticists will need to gather about 400,000 base differences—called single nucleotide polymorphisms—in the human genome to begin to pin down a problematic gene implicated in a disease.

But as Sutter reported at the Cold Spring Harbor meeting, such gene tracking should be much easier in dogs. By incorporating genomic information from 20 dogs from each of five breeds and the previously published poodle sequence (*Science*, 26 September 2003, p. 1898), he calculated that the job can be accomplished with just 30,000 SNPs.

At the same meeting, Lindblad-Toh described her progress sequencing the genome of a boxer named Tasha, chosen because the breed has very little genetic variation. Working with Ostrander and more than two dozen collaborators, Lindblad-Toh has sequenced enough DNA to cover the genome more than seven times over and expects that the consortium will put these data together into a high-quality draft. Once that goes public, which should occur in the next few weeks, finding disease genes in dogs will be even easier.

Dog breeders should be proud.

–ELIZABETH PENNISI

NUMBER THEORY

Proof Promises Progress in Prime Progressions

The theorem that Ben Green and Terence Tao set out to prove would have been impressive enough. Instead, the two mathematicians wound up with a stunning breakthrough in the theory of prime numbers. At least that's the preliminary assessment of experts who are looking at their complicated 50-page proof.

Green, who is currently at the Pacific Institute for the Mathematical Sciences in Vancouver, British Columbia, and Tao of the University of California (UC), Los Angeles, began working 2 years ago on the problem of arithmetic progressions of primes: sequences of primes (numbers divisible only by themselves and 1) that differ by a constant amount. One such sequence is 13, 43, 73, and 103, which differ by 30.

In 1939, Dutch mathematician Johannes van der Corput proved that there are an infinite number of arithmetic progressions of primes with three terms, such as 3, 5, 7 or 31, 37, 43. Green and Tao hoped to prove the same result for four-term progressions. The theorem they got, though, proved the result for prime progressions of *all* lengths.

"It's a very, very spectacular achievement," says Green's former adviser, Timothy Gowers of the University of Cambridge, who received the 1998 Fields Medal, the mathematics equivalent of the Nobel Prize, for work on related problems. Ronald Graham, a combinatorialist at UC San Diego, agrees. "It's just amazing," he says. "It's such a big jump from what came before."

Green and Tao started with a 1975 theorem by Endre Szemerédi of the Hungarian Academy of Sciences. Szemerédi proved that arithmetic progressions of all lengths crop up in any positive fraction of the integers—basically, any subset of integers whose ratio to the whole set doesn't dwindle away to zero as the numbers get larger and larger. The primes don't qualify, because they thin out too rapidly with increasing size. So Green and Tao set out to show that Szemerédi's theorem still holds when the integers are replaced with a smaller set of numbers with special properties, and then to prove that the primes constitute a positive fraction of that set.

To build their set, they applied a branch of mathematics known as ergodic theory (loosely speaking, a theory of mixing or averaging) to mathematical objects called pseudorandom numbers. Pseudorandom numbers are not truly random, because they are generated by rules, but they behave as random numbers do for certain mathematical purposes. Using these tools, Green and Tao constructed a pseudorandom set of primes and "almost primes," numbers with relatively few prime factors compared to their size.

The last step, establishing the primes as a positive fraction of their pseudorandom set, proved elusive. Then Andrew Granville, a number theorist at the University of Montreal, pointed Green to some results by Dan Goldston of San Jose State University in California and Cem Yildirim of Boğaziçi University in Istanbul, Turkey.

2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67 71 73 79 83
89 97 101 103 107 109 113 127 131 137 139 149 151 157 163 167
173 179 181 191 193 197 **199** 211 223 227 229 233 239 241 251
257 263 269 271 277 281 283 293 307 311 313 317 331 337 347
349 353 359 367 373 379 383 389 397 401 **409** 419 421 431 433
439 443 449 457 461 463 467 479 487 491 499 503 509 521 523
541 547 557 563 569 571 577 587 593 599 601 607 613 617 **619**
631 641 643 647 653 659 661 673 677 683 691 701 709 719 727
733 739 743 751 757 761 769 773 787 797 809 811 821 823 827
829 839 853 857 859 863 877 881 883 887 907 911 919 929 937
941 947 953 967 971 977 983 991 997 1009 1013 1019 1021 1031
1033 **1039** 1049 1051 1061 1063 1069 1087 1091 1093 1097 1103
1109 1117 1123 1129 1151 1153 1163 1171 1181 1187 1193 1201
1213 1217 1223 1229 1231 1237 **1249** 1259 1277 1279 1283 1289
1291 1297 1301 1303 1307 1319 1321 1327 1361 1367 1373 1381
1399 1409 1423 1427 1429 1433 1439 1447 1451 1453 **1459** 1471
1481 1483 1487 1489 1493 1499 1511 1523 1531 1543 1549 1553
1559 1567 1571 1579 1583 1597 1601 1607 1609 1613 1619 1621
1627 1637 1657 1663 1667 **1669** 1693 1697 1699 1709 1721 1723
1733 1741 1747 1753 1759 1777 1783 1787 1789 1801 1811 1823
1831 1847 1861 1867 1871 1873 1877 **1879** 1889 1901 1907 1913
1931 1933 1949 1951 1973 1979 1987 1993 1997 1999 2003 2011
2017 2027 2029 2039 2053 2063 2069 2081 2083 2087 **2089** 2099

Prime suspect. Arithmetic progressions such as this 10-prime sequence are infinitely abundant, if a new proof holds up.

Goldston and Yildirim had developed techniques for studying the size of gaps between primes, work that culminated last year in a dramatic breakthrough in the subject—or so they thought. Closer inspection, by Granville among others, undercut their main result (*Science*, 4 April 2003, p. 32; 16 May 2003, p. 1066), although Goldston and Yildirim have since salvaged a less far-ranging finding. But some of the mathematical machinery that these two had set up proved to be tailor-made for Green and Tao's research. "They had actually proven exactly what we needed," Tao says.

The paper, which has been submitted to the *Annals of Mathematics*, is many months from acceptance. "The problem with a quick assessment of it is that it straddles two areas," Granville says. "All of the number theorists who've looked at it feel that the number-theory half is pretty simple and the ergodic theory is daunting, and the ergodic theorists who've looked at it have thought that the ergodic theory is pretty simple and the number theory is daunting."

Even if a mistake does show up, Granville says, "they've certainly succeeded in bringing in new ideas of real import into the subject." And if the proof holds up? "This could be a turning point for analytic number theory," he says.

–BARRY CIPRA

NUMBER THEORY

Third Time Proves Charm for Prime-Gap Theorem

Dan Goldston feels much better. Two years ago the number theorist at San Jose State University in California suffered a discouraging setback. He and Cem Yildirim of Boğaziçi University in Istanbul, Turkey, had announced a dramatic breakthrough in the theory of prime numbers, only to learn that their proof contained a fatal error (*Science*, 4 April 2003, p. 32; 16 May 2003, p. 1066). But now, with the help of János Pintz of the Alfréd Rényi Mathematical Institute in Budapest, Hungary, Goldston and Yildirim have unveiled a new proof of their breakthrough result. This time experts who have examined it say the proof is rock-solid—in part because it is much simpler than the earlier attempt.

"It's of enormous importance," says Brian Conrey, director of the American Institute of Mathematics in Palo Alto, California. "It's going to open the door to lots of stuff." Andrew Granville of the University of Montreal, Quebec, whose work helped torpedo the original flawed proof, agrees. "It's quite a turning point," he says.

Goldston and Yildirim were studying the way one prime number follows another. Prime numbers—positive integers such as 2, 3, 5, 7, 11, and 13, which can't be broken down into smaller factors—become rarer as numbers get larger. On average, the gap between a large prime p and the next prime number is approximately the natural logarithm of p, written $\log p$. But the actual gap between two primes may be far from average. Number theorists long ago proved that there is no upper limit on how large the gap can grow, relative to $\log p$. What Goldston and Yildirim claimed—and, together with Pintz, have now proved—is that the smallest possible gap also continues to shrink relative to $\log p$, as the numbers increase.

The original proof foundered when Granville and Kannan Soundararajan of the University of Michigan, Ann Arbor, spotted a mistake in a single, technical subsection of the proof, known as a lemma. The rest of the proof was fine, and part of it immediately enabled two other mathematicians to make a major breakthrough in studying arithmetic progressions of primes (*Science*, 21 May 2004, p. 1095). Goldston and Yildirim also salvaged a weaker result about prime gaps that improved on previous researchers' work.

Comeback kid. Goldston despaired of rescuing his proof, but a bright idea saved the day.

CREDIT: D. GOLDSTON

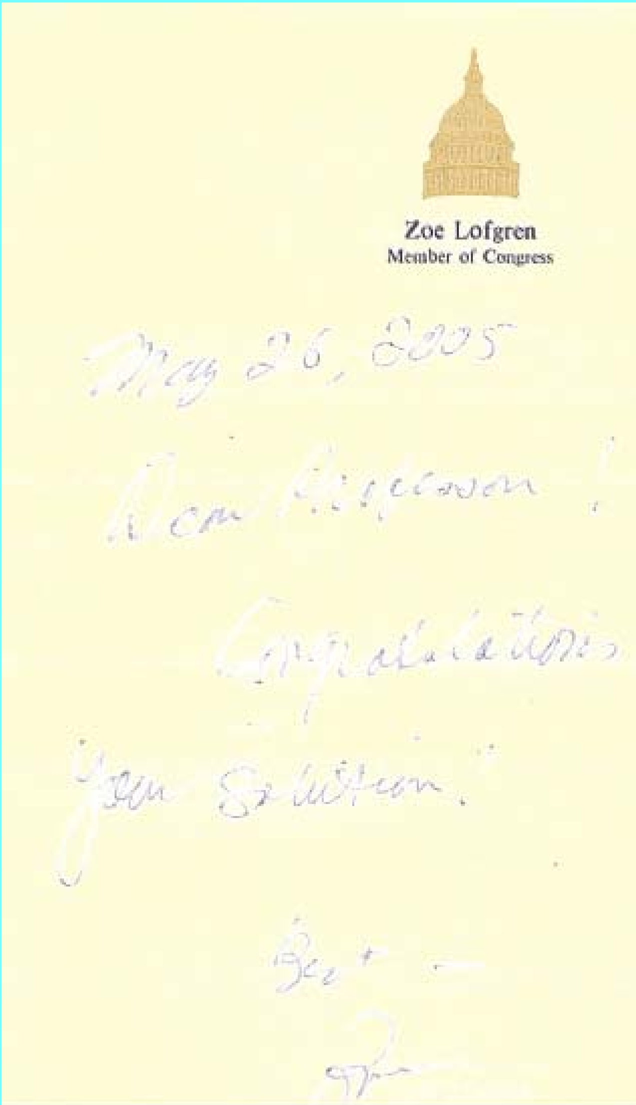
Goldston kept hoping to make the proof work but finally gave up. "I had come to terms with not getting a good result," he recalls. Then, about a year ago, he had an idea for a new approach. He worked out the details and presented his new proof last summer at the mathematical conference center in Oberwohlfach, Germany. He woke up the next morning, however, knowing he had made another mistake, this time in the very last step of the proof. "I really felt jinxed by the whole thing," he recalls.

Pintz, however, took a close look at the flawed proof and came up with the key insight for the ultimate fix. He contacted Goldston and Yildirim last December, and the three number theorists had a complete proof by early February. This time, they were more cautious about announcing the result. "We all thought it was wrong," Goldston says. They circulated the manuscript to a handful of experts, including Granville and Soundararajan, asking them to probe it for any new or remaining errors.

In addition to finding nothing wrong, the ad hoc jury also discovered ways to simplify the proof. "It's been simplified so much there's not much room for an error to be hiding," says Conrey. One of the experts, Yoichi Motohashi of Nihon University in Japan, found a shortcut that led to a surprisingly short proof of the basic, qualitative result. He and the three lead authors have posted this proof, running a mere eight pages, at the arXiv preprint server (www.arxiv.org). The more-detailed paper with Pintz is being rewritten to incorporate some of the simplifications. Goldston gave a public presentation on the new proof at a number theory conference held from 18 to 21 May at the City University of New York.

In itself, the basic result is not a surprise. But it may help mathematicians tackle the famous "twin prime" conjecture, which probably dates back as far as mathematicians have thought about prime numbers. The conjecture holds that there are infinitely many primes for which the gap is 2. The list of twin primes starts with (3, 5), (5, 7), and (11, 13), and has been tabulated by now into the trillions. No one knows whether twin primes ever stop appearing. The new proof is still a far cry from the twin prime conjecture, but it offers a glimmer of hope that number theorists may eventually get there—perhaps a lot sooner than they ever expected. "The twin prime conjecture doesn't seem impossible to prove anymore," Goldston says.

—BARRY CIPRA



Zoe Lofgren
Member of Congress

May 26, 2005

Dear Professor!

Congratulations on your Solution!

Best —

Z

San José State University • Office of the President

June 10, 2005

Dear Professor Goldston,

I enjoyed the article in the San Jose Mercury News recently. Congratulations on your wonderful mathematical accomplishment. I understand that this type of mathematical advance is one of the biggest achievements in years. You should be very proud. I am!

Your hard work, dedication, and I can only imagine, much patience, has finally been rewarded. Again, congratulations on this wonderful discovery.

Sincerely,

Don W. Kassing
President