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Local div. of Heegner points.

1st Century $\underline{1870-1970}$

Kronecker — Shimura

position of CM points

on modular curves

- fields of rationality
- integrality
- reduction mod p (Deuring)

2nd Century: 1970 — ?

Birch, Mazur

Divisors on CM points in Jacobians

$X_0(N)$

$K = \mathbb{Q}(\sqrt{D})$, $D<0$

$p \mid N \Rightarrow p$ split in K (so we get Heegner points)

$(N) = \mathfrak{n}\cdot\bar{\mathfrak{n}}$, $\gcd(\mathfrak{n},\bar{\mathfrak{n}})=1$.

$x \in X_0(N)(\mathbb{C})$

$\left(\mathbb{C}/\mathcal{O}_K \xrightarrow{1} \mathbb{C}/\mathfrak{n}^{-1}\right) \longleftarrow$ Rational over $H = \mathrm{Hilb}(K)$

kernel $\cong \mathfrak{n}^{-1}/\mathcal{O}_K \cong \mathcal{O}_K/\mathfrak{n} \cong \mathbb{Z}/N\mathbb{Z}$.

$$\begin{array}{c} H \\ \big|\ {\scriptstyle Cl(\mathcal{O}_K)} \\ K \\ \big|\ {\scriptstyle \langle 1,\tau\rangle} \\ \mathbb{Q} \end{array}$$

$\tau\sigma\tau^{-1} = \sigma^{-1}$

$\sigma \in Cl(\mathcal{O}_K)$

$$E_D = \sum_\sigma (x^\sigma) - \sum_\sigma (x^{\sigma\tau})$$

divisor of deg 0 supported on CM points defined over K.

$E_D \in \mathrm{Div}^0(X_0(N)/K)^-$

$e_D \in J_0(N)(K)^-$ class of E_D

$$\begin{array}{ccl} J_0(N) & \xrightarrow{\pi} & E \\ \cup\!\!\mid & \nearrow & \\ X_0(N) & & \\ e_D & \longmapsto & P_D \in E(K)^-. \end{array}$$

Global Results:

① P_D has infinite order in $E(K)^-$

$\Updownarrow$ Gross-Zagier

$\Big(L(E/\mathbb{Q},1) \neq 0$ and $L'(E\otimes\chi_D,1) \neq 0.\Big) \Rightarrow L'(E/K,1)\neq 0.$ $\;\; /\!/$

$$\frac{\int_{E(\mathbb{C})} \omega\wedge\bar\omega}{\sqrt{D}} \cdot ht(P_D)$$

② Kolyvagin:

Assume ℓ is a prime

- $\mathbb{Q}(E[\ell]) \cong GL_2(\mathbb{Z}/\ell\mathbb{Z})$
- $\ell \nmid 2\deg(\pi)$ (odd & no congruences?) (what if $\ell \nmid \deg(\pi)$ but $\ell \mid$ congruence modulus?)

Then (a) $\boxed{P_D \text{ is not div by } \ell \text{ in } E(K)^-}$

(b) $E(K)/\ell E(K) \hookrightarrow \mathrm{Sel}(E/K, \ell)$ ~~...~~ cyclic of ord ℓ generated by image of $\bar{P}_D$.

$\bar{P}_D \quad \ell$

hypothesis we want to elim

Suppose p is a prime that is inert in K:

$$P \in E(K)^- \longrightarrow E(K_p)$$

$$K_p \;\Big|_2\; \mathbb{Q}_p$$

Is P_D divisible by ℓ in $E(K_p)^-$?

better assume in this case: $\ell \nmid (p+1+a_p)$

ℓ-divisible

$$0 \longrightarrow E_1 \longrightarrow E(K_p)^- \xrightarrow{\text{reduction}} E(\mathbb{F}_{p^2})^- \longrightarrow 1$$

pro-p group.

Frob_p has an eigenvalue of -1 on $E[\ell]$.

$\text{Frob}(p) = \begin{pmatrix} -1 & 0 \\ 0 & -p \end{pmatrix}$ ← assume $p \not\equiv \pm 1 \pmod{\ell}$

is reduction $P_D \bmod p$ div. by ℓ here

$\tilde{P}_D \in \ell E(\mathbb{F}_{p^2})^-$?

Assume $\text{Frob}_p = \begin{pmatrix} -1 & 0 \\ 0 & p \end{pmatrix}$ on $E[\ell]$

and $p \not\equiv \pm 1 \pmod{\ell}$

$$\Longrightarrow \quad E(\mathbb{F}_{p^2})/\ell = E(\mathbb{F}_{p^2})^-/\ell \cong \mathbb{Z}/\ell\mathbb{Z}$$

$E(q) := E(\mathbb{F}_q)$

When is P_D not div. by ℓ in $E(K_p)^-$ or equiv in $E(p^2)^-$

Ribet: $\exists$ eigenform F of level Np, new at p, with

$$F \equiv \underset{\text{level } N.}{f_E} \pmod{\ell}$$

and $U_p(F) = -F$ (because $\ell \mid (p+1+a_p)$)

Really Ribet defines a max. ideal $\mathfrak{m} \subset \underset{\text{wt 2 level } Np, \text{ new at } p.}{\mathbb{T}}$

$$\mathbb{T}/\mathfrak{m} \cong \mathbb{Z}/\ell\mathbb{Z}$$

$$\begin{tikzcd}
Y \arrow[r, "\text{divisor class}"'] \arrow[rrr, bend left] & J_0(N)(p^2) \arrow[r] & E(p^2) \arrow[r] & E(p^2)/\ell \cong \mathbb{Z}/\ell\mathbb{Z}
\end{tikzcd}$$

Y $\cap$| free abelian group generated by ss points on $X_0(N) \pmod p$ of divisors of degree 0.

Ihara tells cokernel is Shimura subgroup

$\mathfrak{m} = \mathrm{Ann}_{\mathbb{T}}\left(E(p^2)/\ell\right)$

$\mathbb{T}$ acts on Y, in fact $Y \otimes \mathbb{Q}$ is free $\mathbb{T} \otimes \mathbb{Q}$ module of rank 1.

$\mathfrak{m} = \mathrm{Ann}_{\mathbb{T}}($ of $)$

$\mathbb{T}_{\mathfrak{m}}$ acts on $Y \otimes \mathbb{T}_{\mathfrak{m}} \longleftarrow$ free of rank 1 (Emerton)

TFAE:

① $Y \otimes \mathbb{T}_m$ is generated as a free $\mathbb{T}_m$-module of rank 1 by the divison E_D (mod p).

$$E_D = \Sigma(x^{\sigma}) - \Sigma(x^{\sigma\tau})$$

$\Updownarrow$ "just Ihara's theorem"

② P_D is not div. by ℓ in $E(K_p)^-$.

These conditions imply:

$$0 \to A[m] \to A \to A' \to 0$$

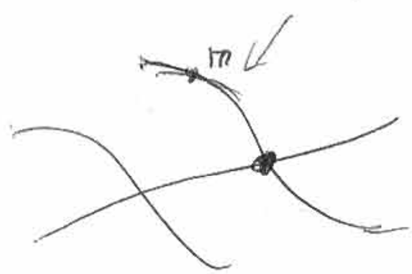
③ $\mathrm{Sel}(A/K, m) = 0$ $\quad = \ker(H^1(K,A) \to \bigoplus_v H^1(K_v, A'))$

where $A \hookrightarrow J_0(Np)$ is the p-new part of $J_0(Np)$

$$\mathbb{T} \hookrightarrow \mathrm{End}(A)$$

<u>Suspect</u>: all equivalent $\longleftarrow$ [Maybe I could for a posi. density of ℓ?]

Assume for simplicity:



Assume $m \subset \mathbb{T}$ comes from an elliptic quotient $A \to E'$, so $\mathrm{Sel}(A/K, m) = \mathrm{Sel}(E', \ell)$.

E of level N

E' of level Np (new at p)

$E[\ell] \cong E'[\ell]$ as repr. of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. $\qquad F = f_{E'} \in \mathbb{Z}[[q]]$.

i.e. $E = -1$
f.e. $E' = +1$ $\Big\} \underset{\text{Nekovar}}{\Rightarrow}$ parities of Selmer are different!

(intersection of these two groups) $\xrightarrow{1 \text{ or } \ell}$ $\mathrm{Sel}(E/_K, \ell)$ $\subset H^1(K, E[\ell])$

$\subset$

(intersection of these two groups) $\xrightarrow{1 \text{ or } \ell}$ $\mathrm{Sel}(E'/_K, \ell)$

containment in one direction with index ℓ, but don't know which way.

local conditions are the same away from p.
but different at p.

$$\begin{array}{ccc} E(K)/_{\ell E(K)} & \hookrightarrow & \mathrm{Sel}(E/_K, \ell) \\ \downarrow & & \downarrow \text{Res} \\ 0 \to E(K_p)/\ell E(K_p) & \hookrightarrow & H^1(K_p, E[\ell]) \end{array}$$

$\mathrm{Frob}(p^2) = \begin{pmatrix} -1 & 0 \\ 0 & -p \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}$

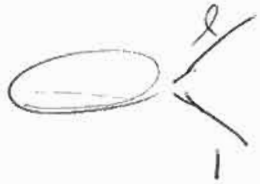
$\mathbb{Z}/_{\ell\mathbb{Z}} \oplus \mu_\ell$.

$$H^1(K_p, \mathbb{Z}/_{\ell\mathbb{Z}}) \oplus H^1(K_p, \mu_\ell)$$
$$\| \qquad\qquad\qquad \|$$
$$\mathbb{Z}/_{\ell\mathbb{Z}} \oplus \mathbb{Z}/_{\ell\mathbb{Z}}$$

image of $E(K_p)$

image of $E'(K_p)$ ← a Tate elliptic curve!

2 $\Longrightarrow$ 3 under hypothesis that P_D is not globally div. by ℓ.



$\mathrm{Sel}(E/K, \ell)$

$\mathrm{Sel}(E'/K, \ell)$

$\therefore$ $\mathrm{Sel}(E'/K, \ell) = 0$ as claimed.

IF P_D ~~not~~ is globally div then

$\mathrm{Sel}(E'/K, \ell)$ at least rank 2.