Math 581e, Fall 2012, Homework 2

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Due: Friday, October 12, 2012

There are 4 problems. Turn your solutions in Friday, October 5, 2012 in class. You may work with other people and can find the LaTeX of this file at http://wstein.org/edu/2012/ant/hw/. Reminder: I normally have office hours 1:00–2:30 on Wednesdays in Padelford C423 (except my Oct 10 office hours will instead be on Oct 8!).

1. Let k be any field. Prove that the polynomial ring $k[t]$ is noetherian. [This problem is very easy if you have been paying attention.]
2. Let k be a field. Let L be a finite degree field extension of the field $k(t)$, i.e., L is a function field. An element $\alpha \in F$ is an algebraic integer if there is some nonzero monic polynomial $f(x) \in k(t)[x]$ with coefficients in $k[t]$ such that $f(\alpha) = 0$.
   (a) Prove that $1/t$ is not an algebraic integer.
   (b) What is the minimial polynomial of $\sqrt{t}+\sqrt{1+t}$? (You may assume $\mathrm{char}(k) \neq 2$ if you want.)
   (c) Prove that the set of algebraic integers in L is a ring.
3. Let $\alpha = \sqrt{2} + \sqrt[3]{3}$. (Use any method to answer this question; even just asking a computer.)
   (a) What is the matrix of multiplication by α on the field $\mathbb{Q}(\sqrt{2}, \sqrt[3]{3})$ with respect to some choice of basis (give the basis you use)?
   (b) What is the trace of α? the norm of α?
   (c) What is the minimial polynomial of α?
4. Let K be a number field and $\alpha \in K$. Prove that the matrix (with respect to some basis) of multiplication by α is diagonalizable (over $\mathbb{C}$).