Homework Assignment 3
Math 252: Modular Abelian Varieties

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Oct. 1 (Due: Oct. 8)

There are four problems, each of equal weight.

1. Let $p \geq 5$ be a prime. In this problem, we apply a general result of Shimura to show that the genus of $X_1(p)$ is $(p-5)(p-7)/24$. Let $\overline{\Gamma}_1(N)$ denote the image of $\Gamma_1(N)$ in $\mathrm{PSL}_2(\mathbf{Z})$.

    (a) Read Prop. 1.40 of Shimura's *Introduction to the Arithmetic Theory of Automorphic Functions*, which asserts that for a finite index subgroup G of $\mathrm{PSL}_2(\mathbf{Z})$, the genus of $X(G)$ is
    $$g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2},$$
    where $\mu = [\mathrm{PSL}_2(\mathbf{Z}) : G]$, ν_2 is the number of elliptic points for G of order 2, ν_3 is the number of elliptic points for G of order 3, and ν_∞ is the number of cusps. An elliptic point of order $n > 1$ is an element $z \in \mathfrak{h}$ whose stabilizer in G has order n. [Hint: You do not have to actually write anything at all to get full credit for this part of the problem!]

    (b) Let $G = \overline{\Gamma}_1(N)$. Show that $\mu = (p^2-1)/2$. [Hint: Use a past homework problem.]

    (c) Let $G = \overline{\Gamma}_1(N)$. Show that $\nu_\infty = p - 1$. [Hint: From a past homework problem, you know that the cusps for $\Gamma(p)$ are in bijection with vectors $\pm(x, y)$, where $x, y, \in \mathbf{Z}/p\mathbf{Z}$ and $\gcd(x, y, p) = 1$. The cusps for $\Gamma_1(p)$ are the orbits of the cusps for $\Gamma(p)$ under the action of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Show that every such orbit contains an element (x, y) with $0 \leq y < p$ and $x = 0$.]

    (d) Let $G = \overline{\Gamma}_1(N)$, with $N \geq 5$. Show that $\nu_2 = \nu_3 = 0$. [Hint: Shimura proves the analogue of this statement in his book. You could find his proof and adapt it.]

    (e) Conclude that if $p \geq 5$ is prime, then the genus of $X_1(p)$ is $(p-5)(p-7)/24$.

2. For $i = 1, 2$, let $g_1 = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \overline{\Gamma}_1(N)$. Then g_1 and g_2 lie in the same right coset $\overline{\Gamma}_1(N)x$ if and only if $c_1 \equiv \varepsilon c_2 \pmod{N}$ and $d_1 \equiv \varepsilon d_2 \pmod{N}$, where $\varepsilon = \pm 1$. [Hint: Compute $g_1 g_2^{-1}$ explicitly to obtain a system of congruences that characterize whether or not g_1 and g_2 are in the same coset.]

3. For $i = 1, 2$ let $\alpha_i = a_i/b_i$ be cusps written in lowest terms. Then $\alpha_2 = \gamma(\alpha_1)$ for some $\gamma \in \overline{\Gamma}_1(N)$ if and only if $b_2 \equiv \varepsilon b_1 \pmod{N}$ and $a_2 \equiv \varepsilon a_1 \pmod{\gcd(b_1, N)}$, with $\varepsilon = \pm 1$. [Hint: The tricky part is to prove that the congruence conditions imply that the cusps are conjugate; for this, construct matrices $\left(\begin{smallmatrix} a_2 & r_2 \\ b_2 & s_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_1 & r_1 \\ b_1 & s_1 \end{smallmatrix}\right)$ that, by Problem 2, lie in the same right coset.]

4. In this problem we compute with modular symbols for the modular curve $X_1(5)$. Recall that
$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$
   (a) Show that the genus of $X_1(5)$ is 0 using Problem 1.
   (b) Use Exercise 2 to write down $(p^2 - 1)/2 = 12$ pairs (c, d) that are the bottom two entries of a set of right coset representatives for $\overline{\Gamma}_1(5)$ in $\mathrm{PSL}_2(\mathbf{Z})$.
   (c) Let C be the free abelian group generated by the pairs (c, d) from part (ii), subject to the relations $x + xs = 0$ and $x = 0$ if $x = xs$, where x runs through the 12 pairs (c, d). What is the rank of C?
   (d) Compute the kernel Z of the linear map $C \to \mathrm{Div}(X_1(5))$ that sends (c, d) to $g(\infty) - g(0) \in \mathrm{Div}(X_1(5))$, where $g = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$.
   (e) Compute the subgroup B of C generated by $x + xt + xt^2$ and also x when $x = xt$, where x runs through the 12 pairs (c, d).
   (f) Observe that $B = Z$, so $\mathrm{H}_1(X_1(5), \mathbf{Z}) = 0$.
   (g) Compute C/B. **Remark:** (not part of the problem) The group C/B is isomorphic to the relative homology
$$\mathrm{H}_1(X_1(5), \mathbf{Z}, \{\text{cusps}\}).$$
   (h) Challenge ("extra credit"): Prove that for any prime p, if we construct C and Z as above, but with $\Gamma_1(5)$ replaced by $\Gamma_1(p)$, then the quotient C/Z is free of rank $p - 2$.