# Homework Assignment 4

(Math 252: Modular Abelian Varieties)

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Oct. 8 (Due: Oct. 15)

There are six problems and parts of problems are of equal weight. (Reminder: It is fine to try to find solutions to these problems in books, etc., and recopy them, as long as you cite your sources.)

1. Let $T = V/L$ be a complex torus. Then we know that the map that sends an element $\varphi \in \mathrm{End}(T) = \mathrm{Hom}(T, T)$ to the corresponding homomorphism $L \to L$ is injective, so there is an injective map $\rho_{\mathbf{Z}} : \mathrm{Hom}(L, L) \approx \mathbf{Z}^{4d}$, where $d = \dim T$. Is it possible that $\rho_Z$ is surjective?

2. Suppose $L_1 = \mathbf{Z} + \mathbf{Z}\alpha i \subset V_1 = \mathbf{C}$, with $\alpha^3 = 2$. Then with respect to the basis $1, \alpha i$, the matrix of complex conjugation is $J_1 = \left(\begin{smallmatrix} 0 & -\alpha \\ 1/\alpha & 0 \end{smallmatrix}\right)$. Use this to prove that $\mathrm{End}(V_1/L_1) = \mathbf{Z}$.

3. Let $L$ be a lattice in a vector space $V$. A subgroup $M$ of $L$ is *saturated in* $L$ if $L/M$ is torsion free.
   (a) Suppose $W$ is a vector space and $f : L \to W$ is a homomorphism. Prove that $\ker(L)$ is saturated in $L$.
   (b) Suppose $W \subset V$ is a subspace of $V$. Prove that $L \cap W$ is saturated in $L$.
   (c) Suppose $M$ is a subgroup of $L$. The saturation $M'$ of $M$ in $L$ is the intersection of $\mathbf{Q}M$ with $L$. Prove that $M$ has finite index in $M'$.

4. Let $T = V/L$ be a complex torus and let $\rho_{\mathbf{Z}} : \mathrm{End}(T) \to \mathrm{End}_{\mathbf{Z}}(L)$ be the integral representation. Prove that the image of $\rho_{\mathbf{Z}}$ is saturated in $\mathrm{End}_{\mathbf{Z}}(L)$, in the sense that the cokernel of $\rho_{\mathbf{Z}}$ is torsion free.

5. (a) Let $T = V/L$ be a complex torus. Prove that as a real manifold $T$ is isomorphic to a product of copies of $S^1 = \mathbf{R}/\mathbf{Z}$. Thus topologically complex tori are boring since they are classified by their dimension; it is their complex structure that makes them interesting.
   (b) (*) Construct a 2-dimensional complex torus $T = V/L$ (so $V$ is a 2-dimensional complex vector space), such that $T$ is not isomorphic as a complex torus to a product $\mathbf{C}/L_1 \times \mathbf{C}/L_2$ of 1-dimensional complex tori.

6. Let $L$ be the subgroup of $V = \mathbf{C} \times \mathbf{C}$ generated by $(1, 0)$, $(0, i)$, $(1, \sqrt{2})$, $(i, \sqrt{2})$.
   (a) Prove that $L$ is a lattice in $V$.
   (b) With respect to the given basis for $L$, compute the matrix $J$ that represents multiplication by $i$ from $V$ to $V$.
   (c) (*) Let $T = V/L$. Compute $\mathrm{End}(T)$, which is the subgroup of $\mathrm{End}_{\mathbf{Z}}(L, L)$ of elements that commute with $J$.
   (d) (*) Find a complex torus $T'$ that is isogenous to $T$ but not isomorphic to $T$.