1.3 Abelian Varieties as Complex Tori (10/08/03 notes for Math 252 by William Stein)

In this section we introduce extra structure on a complex torus $T = V/L$ that will enable us to understand whether or not T is isomorphic to $A(\mathbf{C})$, for some abelian variety A over $\mathbf{C}$. When $\dim T = 1$, the theory of the Weierstrass $\wp$ function implies that T is always $E(\mathbf{C})$ for some elliptic curve. In contrast, the generic torus of dimension > 1 does not arise from an abelian variety.

In this section we introduce the basic structures on complex tori that are needed to understand which tori arise from abelian varieties, to construct the dual of an abelian variety, to see that $\mathrm{End}_0(A)$ is a semisimple $\mathbf{Q}$-algebra, and to understand the polarizations on an abelian variety. For proofs, including extensive motivation from the one-dimensional case, read the beautifully written book [4] by Swinnerton-Dyer, and for another survey that strongly influenced the discussion below, see Rosen's [3].

1.3.1 Hermitian and Riemann Forms

Let V be a finite-dimensional complex vector space.

Definition 1.3.1 (Hermitian form). A *Hermitian form* is a conjugate-symmetric pairing

$$H : V \times V \to \mathbf{C}$$

that is $\mathbf{C}$-linear in the first variable and $\mathbf{C}$-antilinear in the second. Thus H is $\mathbf{R}$-bilinear, $H(iu, v) = iH(u, v) = H(u, \bar{i}v)$, and $H(u, v) = \overline{H(v, u)}$.

Write $H = S + iE$, where $S, E : V \times V \to \mathbf{R}$ are real bilinear pairings.

Proposition 1.3.2. *Let H, S, and E be as above.*

1. *We have that S is symmetric, E is antisymmetric, and*

$$S(u, v) = E(iu, v), \quad S(iu, iv) = S(u, v), \quad E(iu, iv) = E(u, v).$$

2. *Conversely, if E is a real-valued antisymmetric bilinear pairing on V such that $E(iu, iv) = E(u, v)$, then $H(u, v) = E(iu, v) + iE(u, v)$ is a Hermitian form on V. Thus there is a bijection between the Hermitian forms on V and the real, antisymmetric bilinear forms E on V such that $E(iu, iv) = E(u, v)$.*

Proof. To see that S is symmetric, note that $2S = H + \overline{H}$ and $H + \overline{H}$ is symmetric because H is conjugate symmetric. Likewise, $E = (H - \overline{H})/(2i)$, so

$$E(v, u) = \frac{1}{2i}\left(H(v, u) - \overline{H(v, u)}\right) = \frac{1}{2i}\left(\overline{H(u, v)} - H(u, v)\right) = -E(u, v),$$

which implies that E is antisymmetric. To see that $S(u, v) = E(iu, v)$, rewrite both $S(u, v)$ and $E(iu, v)$ in terms of H and simplify to get an identity. The other two identities follow since

$$H(iu, iv) = iH(u, iv) = i\bar{i}H(u, v) = H(u, v).$$

Suppose $E : V \times V \to \mathbf{R}$ is as in the second part of the proposition. Then

$$H(iu, v) = E(i^2u, v) + iE(iu, v) = -E(u, v) + iE(iu, v) = iH(u, v),$$

and the other verifications of linearity and antilinearity are similar. For conjugate symmetry, note that

$$\begin{aligned} H(v,u) &= E(iv,u) + iE(v,u) = -E(u,iv) - iE(u,v) \\ &= -E(iu,-v) - iE(u,v) = H(u,v). \end{aligned}$$

□

Note that the set of Hermitian forms is a group under addition.

Definition 1.3.3 (Riemann form). A *Riemann form* on a complex torus $T = V/L$ is a Hermitian form H on V such that the restriction of $E = \mathrm{Im}(H)$ to L is integer valued. If $H(u,u) \geq 0$ for all $u \in V$ then H is *positive semi-definite* and if H is positive and $H(u,u) = 0$ if and only if $u = 0$, then H is *nondegenerate*.

Theorem 1.3.4. *Let T be a complex torus. Then T is isomorphic to $A(\mathbf{C})$, for some abelian variety A, if and only if there is a nondegenerate Riemann form on T.*

This is a nontrivial theorem, which we will not prove here. It is proved in [4, Ch.2] by defining an injective map from positive divisors on $T = V/L$ to positive semi-definite Riemann forms, then constructing positive divisors associated to theta functions on V. If H is a nondegenerate Riemann form on T, one computes the dimension of a space of theta functions that corresponds to H in terms of the determinant of $E = \mathrm{Im}(H)$. Since H is nondegenerate, this space of theta functions is nonzero, so there is a corresponding nondegenerate positive divisor D. Then a basis for

$$L(3D) = \{f : (f) + 3D \text{ is positive }\} \cup \{0\}$$

determines an embedding of T in a projective space.

Why the divisor $3D$ instead of D above? For an elliptic curve $y^2 = x^3 + ax + b$, we could take D to be the point at infinity. Then $L(3D)$ consists of the functions with a pole of order at most 3 at infinity, which contains 1, x, and y, which have poles of order 0, 2, and 3, respectively.

Remark 1.3.5. (Copied from page 39 of [4].) When $n = \dim V > 1$, however, a general lattice L will admit no nonzero Riemann forms. For if $\lambda_1, \ldots, \lambda_{2n}$ is a base for L then E as an $\mathbf{R}$-bilinear alternating form is uniquely determined by the $E(\lambda_i, \lambda_j)$, which are integers; and the condition $E(z,w) = E(iz,iw)$ induces linear relations with real coefficients between $E(\lambda_i, \lambda_j)$, which for general L have no nontrivial integer solutions.

1.3.2 Complements, Quotients, and Semisimplicity of the Endomorphism Algebra

Lemma 1.3.6. *If T possesses a nondegenerate Riemann form and $T' \subset T$ is a subtorus, then T' also possesses a nondegenerate Riemann form.*

Proof. If H is a nondegenerate Riemann form on a torus T and T' is a subtorus of T, then the restriction of H to T' is a nondegenerate Riemann form on T' (the restriction is still nondegenerate because H is positive definite). □

Lemma 1.3.6 and Lemma 1.2.3 together imply that the kernel of a homomorphism of abelian varieties is an extension of an abelian variety by a finite group.

Lemma 1.3.7. *If T possesses a nondegenerate Riemann form and $T \to T'$ is an isogeny, then T' also possesses a nondegenerate Riemann form.*

Proof. Suppose $T = V/L$ and $T' = V'/L'$. Since the isogeny is induced by an isomorphism $V \to V'$ that sends L into L', we may assume for simplicity that $V = V'$ and $L \subset L'$. If H is a nondegenerate Riemann form on V/L, then $E = \mathrm{Re}(H)$ need not be integer valued on L'. However, since L has finite index in L', there is some integer d so that dE is integer valued on L'. Then dH is a nondegenerate Riemann form on V/L'. □

Note that Lemma 1.3.7 implies that the quotient of an abelian variety by a finite subgroup is again an abelian variety.

Theorem 1.3.8 (Poincare Reducibility). *Let A be an abelian variety and suppose $A' \subset A$ is an abelian subvariety. Then there is an abelian variety $A'' \subset A$ such that $A = A' + A''$ and $A' \cap A''$ is finite. (Thus A is isogenous to $A' \times A''$.)*

Proof. We have $A(\mathbf{C}) \approx V/L$ and there is a nondegenerate Riemann form H on V/L. The subvariety A' is isomorphic to V'/L', where V' is a subspace of V and $L' = V' \cap L$. Let V'' be the orthogonal complement of V' with respect to H, and let $L'' = L \cap V''$. To see that L'' is a lattice in V'', it suffices to show that L'' is the orthogonal complement of L' in L with respect to $E = \mathrm{Im}(H)$, which, because E is integer valued, will imply that L'' has the correct rank. First, suppose that $v \in L''$; then, by definition, v is in the orthogonal complement of L' with respect to H, so for any $u \in L'$, we have $0 = H(u, v) = S(u, v) + iE(u, v)$, so $E(u, v) = 0$. Next, suppose that $v \in L$ satisfies $E(u, v) = 0$ for all $u \in L'$. Since $V' = \mathbf{R}L'$ and E is $\mathbf{R}$-bilinear, this implies $E(u, v) = 0$ for any $u \in V'$. In particular, since V' is a complex vector space, if $u \in L'$, then $S(u, v) = E(iu, v) = 0$, so $H(u, v) = 0$.

We have shown that L'' is a lattice in V'', so $A'' = V''/L''$ is an abelian subvariety of A. Also $L' + L''$ has finite index in L, so there is an isogeny $V'/L' \oplus V''/L'' \to V/L$ induced by the natural inclusions. □

Proposition 1.3.9. *Suppose $A' \subset A$ is an inclusion of abelian varieties. Then the quotient A/A' is an abelian variety.*

Proof. Suppose $A = V/L$ and $A' = V'/L'$, where V' is a subspace of V. Let $W = V/V'$ and $M = L/(L \cap V')$. Then, W/M is isogenous to the complex torus V''/L'' of Theorem 1.3.8 via the natural map $V'' \to W$. Applying Lemma 1.3.7 completes the proof. □

Definition 1.3.10. An abelian variety A is *simple* if it has no nonzero proper abelian subvarieties.

Proposition 1.3.11. *The algebra* $\mathrm{End}_0(A)$ *is semisimple.*

Proof. Using Theorem 1.3.8 and induction, we can find an isogeny

$$A \simeq A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_r^{n_r}$$

with each A_i simple. Since $\mathrm{End}_0(A) = \mathrm{End}(A) \otimes \mathbf{Q}$ is unchanged by isogeny, and $\mathrm{Hom}(A_i, A_j) = 0$ when $i \neq j$, we have

$$\mathrm{End}_0(A) = \mathrm{End}_0(A_1^{n_1}) \times \mathrm{End}_0(A_2^{n_2}) \times \cdots \times \mathrm{End}_0(A_r^{n_r})$$

Each of $\mathrm{End}_0(A_i^{n_i})$ is isomorphic to $M_{n_i}(D_i)$, where $D_i = \mathrm{End}_0(A_i)$. By Schur's Lemma, $D_i = \mathrm{End}_0(A_i)$ is a division algebra over $\mathbf{Q}$ (proof: any nonzero endomorphism has trivial kernel, and any injective linear transformation of a $\mathbf{Q}$-vector space is invertible), so $\mathrm{End}_0(A)$ is a product of matrix algebras over division algebras over $\mathbf{Q}$, which proves the proposition. □

1.3.3 Theta Functions

Suppose $T = V/L$ is a complex torus.

Definition 1.3.12 (Theta function). Let $M : V \times L \to C$ and $J : L \to \mathbf{C}$ be set-theoretic maps such that for each $\lambda \in L$ the map $z \mapsto M(z, \lambda)$ is $\mathbf{C}$-linear. A *theta function* of type (M, J) is a function $\theta : V \to \mathbf{C}$ such that for all $z \in V$ and $\lambda \in L$, we have

$$\theta(z + \lambda) = \theta(z) \cdot \exp(2\pi i(M(z, \lambda) + J(\lambda))).$$

Suppose that $\theta(z)$ is a nonzero holomorphic theta function of type (M, J). The $M(z, \lambda)$, for various λ, cannot be unconnected. Let $F(z, \lambda) = 2\pi i(M(z, \lambda) + J(\lambda))$.

Lemma 1.3.13. *For any $\lambda, \lambda' \in L$, we have*

$$F(z, \lambda + \lambda') = F(z + \lambda, \lambda') + F(z, \lambda) \pmod{2\pi i}.$$

Thus

$$M(z, \lambda + \lambda') = M(z, \lambda) + M(z, \lambda'), \tag{1.3.1}$$

and

$$J(\lambda + \lambda') - J(\lambda) - J(\lambda') \equiv M(\lambda, \lambda') \pmod{\mathbf{Z}}.$$

Proof. Page 37 of [4]. □

Using (1.3.1) we see that M extends uniquely to a function $\tilde{M} : V \times V \to \mathbf{C}$ which is $\mathbf{C}$-linear in the first argument and $\mathbf{R}$-linear in the second. Let

$$E(z, w) = \tilde{M}(z, w) - M(w, z),$$

$$H(z, w) = E(iz, w) + iE(z, w).$$

Proposition 1.3.14. *The pairing H is Riemann form on T with real part E.*

We call H the Riemann form associated to θ.

1.3.4 Example: Complex Tori that are not Abelian Varieties

1.4 Divisors and the Néron-Severi and Picard Groups

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