14

Hecke operators as correspondences:

W. Stein, Math 252, 10/31/03

Our goal is to view the Hecke operators T_n and $\langle d \rangle$ as objects defined over $\mathbf{Q}$ that act in a compatible way on modular forms, modular Jacobians, and homology. In order to do this, we will define the Hecke operators as correspondences.

14.1 The Definition

Definition 14.1.1 (Correspondence). Let C_1 and C_2 be curves. A *correspondence* $C_1 \rightsquigarrow C_2$ is a curve C together with nonconstant morphisms $\alpha : C \to C_1$ and $\beta : C \to C_2$. We represent a correspondence by a diagram

$$\begin{array}{ccc} & C & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ C_1 & & C_2 \end{array}$$

Given a correspondence $C_1 \rightsquigarrow C_2$ the *dual correspondence* $C_2 \rightsquigarrow C_1$ is obtained by looking at the diagram in a mirror

$$\begin{array}{ccc} & C & \\ {}^{\beta}\swarrow & & \searrow^{\alpha} \\ C_2 & & C_1 \end{array}$$

In defining Hecke operators, we will focus on the simple case when the modular curve is $X_0(N)$ and Hecke operator is T_p, where $p \nmid N$. We will view T_p as a correspondence $X_0(N) \rightsquigarrow X_0(N)$, so there is a curve $C = X_0(pN)$ and maps α and β fitting into a diagram

$$\begin{array}{ccc} & X_0(pN) & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ X_0(N) & & X_0(N). \end{array}$$

The maps α and β are degeneracy maps which forget data. To define them, we view $X_0(N)$ as classifying isomorphism classes of pairs (E, C), where E is an elliptic curve and C is a cyclic subgroup of order N (we will not worry about what happens at the cusps, since any rational map of nonsingular curves extends uniquely to a morphism). Similarly, $X_0(pN)$ classifies isomorphism classes of pairs (E, G) where $G = C \oplus D$, C is cyclic of order N and D is cyclic of order p. Note that since $(p, N) = 1$, the group G is cyclic of order pN and the subgroups C and D are uniquely determined by G. The map α forgets the subgroup D of order p, and β quotients out by D:

$$\alpha : (E, G) \mapsto (E, C) \tag{14.1.1}$$

$$\beta : (E, G) \mapsto (E/D, (C + D)/D) \tag{14.1.2}$$

We translate this into the language of complex analysis by thinking of $X_0(N)$ and $X_0(pN)$ as quotients of the upper half plane. The first map α corresponds to the map

$$\Gamma_0(pN)\backslash\mathfrak{h} \to \Gamma_0(N)\backslash\mathfrak{h}$$

induced by the inclusion $\Gamma_0(pN) \hookrightarrow \Gamma_0(N)$. The second map β is constructed by composing the isomorphism

$$\Gamma_0(pN)\backslash\mathfrak{h} \xrightarrow{\sim} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(pN) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \backslash\mathfrak{h} \tag{14.1.3}$$

with the map to $\Gamma_0(N)\backslash\mathfrak{h}$ induced by the inclusion

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(pN) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \subset \Gamma_0(N).$$

The isomorphism (14.1.3) is induced by $z \mapsto \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) z = pz$; explicitly, it is

$$\Gamma_0(pN)z \mapsto \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \Gamma_0(pN) \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) z.$$

(Note that this is well-defined.)

The maps α and β induce pullback maps on differentials

$$\alpha^*, \beta^* : \mathrm{H}^0(X_0(N), \Omega^1) \to \mathrm{H}^0(X_0(pN), \Omega^1).$$

We can identify $S_2(\Gamma_0(N))$ with $H^0(X_0(N), \Omega^1)$ by sending the cusp form $f(z)$ to the holomorphic differential $f(z)dz$. Doing so, we obtain two maps

$$\alpha^*, \beta^* : S_2(\Gamma_0(N)) \to S_2(\Gamma_0(pN)).$$

Since α is induced by the identity map on the upper half plane, we have $\alpha^*(f) = f$, where we view $f = \sum a_n q^n$ as a cusp form with respect to the smaller group $\Gamma_0(pN)$. Also, since β^* is induced by $z \mapsto pz$, we have

$$\beta^*(f) = p \sum_{n=1}^{\infty} a_n q^{pn}.$$

The factor of p is because

$$\beta^*(f(z)dz) = f(pz)d(pz) = pf(pz)dz.$$

Let X, Y, and C be curves, and α and β be nonconstant holomorphic maps, so we have a correspondence

$$X \xleftarrow{\ \alpha\ } C \xrightarrow{\ \beta\ } Y.$$

By first pulling back, then pushing forward, we obtain induced maps on differentials

$$H^0(X,\Omega^1) \xrightarrow{\alpha^*} H^0(C,\Omega^1) \xrightarrow{\beta_*} H^0(Y,\Omega^1).$$

The composition $\beta_* \circ \alpha^*$ is a map $H^0(X,\Omega^1) \to H^0(Y,\Omega^1)$. If we consider the dual correspondence, which is obtained by switching the roles of X and Y, we obtain a map $H^0(Y,\Omega^1) \to H^0(X,\Omega^1)$.

Now let α and β be as in (14.1.1). Then we can recover the action of T_p on modular forms by considering the induced map

$$\beta_* \circ \alpha^* : H^0(X_0(N),\Omega^1) \to H^0(X_0(N),\Omega^1)$$

and using that $S_2(\Gamma_0(N)) \cong H^0(X_0(N),\Omega^1)$.

14.2 Maps induced by correspondences

In this section we will see how correspondences induce maps on divisor groups, which in turn induce maps on Jacobians.

Suppose $\varphi : X \to Y$ is a morphism of curves. Let $\Gamma \subset X \times Y$ be the graph of φ. This gives a correspondence

$$X \xleftarrow{\ \alpha\ } \Gamma \xrightarrow{\ \beta\ } Y$$

We can reconstruct φ from the correspondence by using that $\varphi(x) = \beta(\alpha^{-1}(x))$. [draw picture here]

More generally, suppose Γ is a curve and that $\alpha : \Gamma \to X$ has degree $d \geq 1$. View $\alpha^{-1}(x)$ as a divisor on Γ (it is the formal sum of the points lying over x, counted with appropriate multiplicities). Then $\beta(\alpha^{-1}(x))$ is a divisor on Y. We thus obtain a map

$$\mathrm{Div}^n(X) \xrightarrow{\beta\circ\alpha^{-1}} \mathrm{Div}^{dn}(Y),$$

where $\mathrm{Div}^n(X)$ is the group of divisors of degree n on X. In particular, setting $d = 0$, we obtain a map $\mathrm{Div}^0(X) \to \mathrm{Div}^0(Y)$.

We now apply the above construction to T_p. Recall that T_p is the correspondence

$$X_0(N) \xleftarrow{\ \alpha\ } X_0(pN) \xrightarrow{\ \beta\ } X_0(N),$$

where α and β are as in Section 14.1 and the induced map is

$$(E, C) \overset{\alpha^*}{\mapsto} \sum_{D \in E[p]} (E, C \oplus D) \overset{\beta_*}{\mapsto} \sum_{D \in E[p]} (E/D, (C+D)/D).$$

Thus we have a map $\mathrm{Div}(X_0(N)) \to \mathrm{Div}(X_0(N))$. This strongly resembles the first definition we gave of T_p on level 1 forms, where T_p was a correspondence of lattices.

14.3 Induced maps on Jacobians of curves

Let X be a curve of genus g over a field k. Recall that there is an important association

$$\Big\{ \text{ curves } X/k \Big\} \longrightarrow \Big\{ \text{ Jacobians } \mathrm{Jac}(X) = J(X) \text{ of curves } \Big\}$$

between curves and their Jacobians.

Definition 14.3.1 (Jacobian). Let X be a curve of genus g over a field k. Then the *Jacobian* of X is an abelian variety of dimension g over k whose underlying group is functorially isomorphic to the group of divisors of degree 0 on X modulo linear equivalence. (For a more precise definition, see Section **??** (Jacobians section).)

There are many constructions of the Jacobian of a curve. We first consider the Albanese construction. Recall that over $\mathbf{C}$, any abelian variety is isomorphic to $\mathbf{C}^g/L$, where L is a lattice (and hence a free $\mathbf{Z}$-module of rank $2g$). There is an embedding

$$\iota : \mathrm{H}_1(X, \mathbf{Z}) \hookrightarrow \mathrm{H}^0(X, \Omega^1)^*$$
$$\gamma \mapsto \int_\gamma \bullet$$

Then we realize $\mathrm{Jac}(X)$ as a quotient

$$\mathrm{Jac}(X) = \mathrm{H}^0(X, \Omega^1)^*/\iota(\mathrm{H}_1(X, \mathbf{Z})).$$

In this construction, $\mathrm{Jac}(X)$ is most naturally viewed as covariantly associated to X, in the sense that if $X \to Y$ is a morphism of curves, then the resulting map $\mathrm{H}^0(X, \Omega^1)^* \to \mathrm{H}^0(Y, \Omega^1)^*$ on tangent spaces induces a map $\mathrm{Jac}(X) \to \mathrm{Jac}(Y)$.

There are other constructions in which $\mathrm{Jac}(X)$ is contravariantly associated to X. For example, if we view $\mathrm{Jac}(X)$ as $\mathrm{Pic}^0(X)$, and $X \to Y$ is a morphism, then pullback of divisor classes induces a map $\mathrm{Jac}(Y) = \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X) = \mathrm{Jac}(X)$.

If $F : X \rightsquigarrow Y$ is a correspondence, then F induces an a map $\mathrm{Jac}(X) \to \mathrm{Jac}(Y)$ and also a map $\mathrm{Jac}(Y) \to \mathrm{Jac}(X)$. If $X = Y$, so that X and Y are the same, it can often be confusing to decide which duality to use. Fortunately, for T_p, with p prime to N, it does not matter which choice we make. But it matters a lot if $p \mid N$ since then we have non-commuting confusable operators and this has resulted in mistakes in the literature.