12
Newforms

These are notes for Math 252 by William Stein. These are based on lectures given by Ken Ribet at Berkeley in 1996.

First we discuss explicitly how U_p, for $p \mid N$, acts on old forms, and how U_p can fail to be diagonalizable. Then we describe a canonical generator for $S_k(\Gamma_1(N))$ as a free module over $\mathbf{T_C}$. Finally, we observe that the subalgebra of $\mathbf{T_Q}$ generated by Hecke operators T_n with $(n, N) = 1$ is isomorphic to a product of number fields.

12.1 The U_p Operator

Let N be a positive integer and M a divisor of N. For each divisor d of N/M we define a map

$$\alpha_d : S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N)) : \quad f(\tau) \mapsto f(d\tau).$$

We verify that $f(d\tau) \in S_k(\Gamma_1(N))$ as follows. Recall that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we write

$$(f|[\gamma]_k)(\tau) = \det(\gamma)^{k-1}(cz+d)^{-k} f(\gamma(\tau)).$$

The transformation condition for f to be in $S_k(\Gamma_1(N))$ is that $f|[\gamma]_k(\tau) = f(\tau)$. Let $f(\tau) \in S_k(\Gamma_1(M))$ and let $\iota_d = \left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $f|[\iota_d]_k(\tau) = d^{k-1} f(d\tau)$ is a modular form on $\Gamma_1(N)$ since $\iota_d^{-1}\Gamma_1(M)\iota_d$ contains $\Gamma_1(N)$. Moreover, if f is a cusp form then so is $f|[\iota_d]_k$.

Proposition 12.1.1. *If $f \in S_k(\Gamma_1(M))$ is nonzero, then*

$$\left\{ \alpha_d(f) \, : \, d \mid \frac{N}{M} \right\}$$

is linearly independent.

Proof. If the q-expansion of f is $\sum a_n q^n$, then the q-expansion of $\alpha_d(f)$ is $\sum a_n q^{dn}$. The matrix of coefficients of the q-expansions of $\alpha_d(f)$, for $d \mid (N/M)$, is upper triangular. Thus the q-expansions of the $\alpha_d(f)$ are linearly independent, hence the $\alpha_d(f)$ are linearly independent, since the map that sends a cusp form to its q-expansion is linear. □

When $p \mid N$, we denote by U_p the Hecke operator T_p acting on the image space $S_k(\Gamma_1(N))$. For $f = \sum a_n q^n \in S_k(\Gamma_1(N))$, we have

$$f|U_p = \sum a_{np} q^n.$$

Suppose $f = \sum a_n q^n \in S_k(\Gamma_1(M))$ is a normalized eigenform for all of the Hecke operators T_n and $\langle n \rangle$, and p is a prime that does not divide M. Then

$$f|T_p = a_p f \quad \text{and} \quad f|\langle p \rangle = \varepsilon(p) f.$$

Assume $N = p^r M$, where $r \geq 1$ is an integer. Let

$$f_i(\tau) = f(p^i \tau),$$

so $f_0, \ldots, f_r$ are the images of f under the maps $\alpha_{p^0}, \ldots, \alpha_{p^r}$, respectively, and $f = f_0$. We have

$$\begin{aligned} f|T_p &= \sum_{n \geq 1} a_{np} q^n + \varepsilon(p) p^{k-1} \sum a_n q^{pn} \\ &= f_0|U_p + \varepsilon(p) p^{k-1} f_1, \end{aligned}$$

so

$$f_0|U_p = f|T_p - \varepsilon(p)p^{k-1} f_1 = a_p f_0 - \varepsilon(p) p^{k-1} f_1.$$

Also

$$f_1|U_p = \left(\sum a_n q^{pn}\right)|U_p = \sum a_n q^n = f_0.$$

More generally, for any $i \geq 1$, we have $f_i|U_p = f_{i-1}$.

The operator U_p preserves the two dimensional vector space spanned by f_0 and f_1, and the matrix of U_p with respect to the basis f_0, f_1 is

$$A = \begin{pmatrix} a_p & 1 \\ -\varepsilon(p)p^{k-1} & 0 \end{pmatrix},$$

which has characteristic polynomial

$$X^2 - a_p X + p^{k-1}\varepsilon(p). \tag{12.1.1}$$

12.1.1 A Connection with Galois Representations

This leads to a striking connection with Galois representations. Let f be a newform and let $K = K_f$ be the field generated over $\mathbf{Q}$ by the Fourier coefficients of f. Let ℓ be a prime and λ a prime lying over ℓ. Then Deligne (and Serre, when $k = 1$) constructed a representation

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}(2, K_\lambda).$$

If $p \nmid N\ell$, then ρ_λ is unramified at p, so if $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ if a Frobenius element, then $\rho_\lambda(\mathrm{Frob}_p)$ is well defined, up to conjugation. Moreover, one can show that

$$\begin{aligned} \det(\rho_\lambda(\mathrm{Frob}_p)) &= p^{k-1}\varepsilon(p), \quad \text{and} \\ \mathrm{tr}(\rho_\lambda(\mathrm{Frob}_p)) &= a_p. \end{aligned}$$

(We will discuss the proof of these relations further in the case $k = 2$.) Thus the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_p) \in \mathrm{GL}_2(E_\lambda)$ is

$$X^2 - a_p X + p^{k-1}\varepsilon(p),$$

which is the same as (12.1.1).

12.1.2 When is U_p Semisimple?

Question 12.1.2. Is U_p semisimple on the span of f_0 and f_1?

If the eigenvalues of U_p are distinct, then the answer is yes. If the eigenvalues are the same, then $X^2 - a_p X + p^{k-1}\varepsilon(p)$ has discriminant 0, so $a_p^2 = 4p^{k-1}\varepsilon(p)$, hence

$$a_p = 2p^{\frac{k-1}{2}}\sqrt{\varepsilon(p)}.$$

Open Problem 12.1.3. Does there exist an eigenform $f = \sum a_n q^n \in S_k(\Gamma_1(N))$ such that $a_p = 2p^{\frac{k-1}{2}}\sqrt{\varepsilon(p)}$?

It is a curious fact that the Ramanujan conjectures, which were proved by Deligne in 1973, imply that $|a_p| \leq 2p^{(k-1)/2}$, so the above equality remains taunting. When $k = 2$, Coleman and Edixhoven proved that $|a_p| < 2p^{(k-1)/2}$.

12.1.3 An Example of Non-semisimple U_p

Suppose $f = f_0$ is a normalized eigenform. Let W be the space spanned by f_0, f_1 and let V be the space spanned by f_0, f_1, f_2, f_3. Then U_p acts on V/W by $\overline{f}_2 \mapsto 0$ and $\overline{f}_3 \mapsto \overline{f}_2$. Thus the matrix of the action of U_p on V/W is $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, which is nonzero and nilpotent, hence not semisimple. Since W is invariant under U_p this shows that U_p is not semisimple on V, i.e., U_p is not diagonalizable.

12.2 The Cusp Forms are Free of Rank One over $\mathbf{T_C}$

12.2.1 Level 1

Suppose $N = 1$, so $\Gamma_1(N) = \mathrm{SL}_2(\mathbf{Z})$. Using the Petersson inner product, we see that all the T_n are diagonalizable, so $S_k = S_k(\Gamma_1(1))$ has a basis

$$f_1, \ldots, f_d$$

of normalized eigenforms where $d = \dim S_k$. This basis is canonical up to ordering. Let $\mathbf{T_C} = \mathbf{T} \otimes \mathbf{C}$ be the ring generated over $\mathbf{C}$ by the Hecke operator T_p. Then, having fixed the basis above, there is a canonical map

$$\mathbf{T_C} \hookrightarrow \mathbf{C}^d : \quad T \mapsto (\lambda_1, \ldots, \lambda_d),$$

where $f_i | T = \lambda_i f_i$. This map is injective and $\dim \mathbf{T_C} = d$, so the map is an isomorphism of $\mathbf{C}$-vector spaces.

The form

$$v = f_1 + \cdots + f_n$$

generates S_k as a $\mathbf{T}$-module. Note that v is canonical since it does not depend on the ordering of the f_i. Since v corresponds to the vector $(1, \ldots, 1)$ and $\mathbf{T} \cong \mathbf{C}^d$ acts on $S_k \cong \mathbf{C}^d$ componentwise, this is just the statement that $\mathbf{C}^d$ is generated by $(1, \ldots, 1)$ as a $\mathbf{C}^d$-module.

There is a perfect pairing $S_k \times \mathbf{T_C} \to \mathbf{C}$ given by

$$\left\langle \sum f, T_n \right\rangle = a_1(f | T_n) = a_n(f),$$

where $a_n(f)$ denotes the nth Fourier coefficient of f. Thus we have simultaneously:

1. S_k is free of rank 1 over $\mathbf{T}_{\mathbf{C}}$, and
2. $S_k \cong \mathrm{Hom}_{\mathbf{C}}(\mathbf{T}_{\mathbf{C}}, \mathbf{C})$ as $\mathbf{T}$-modules.

Combining these two facts yields an isomorphism

$$\mathbf{T}_{\mathbf{C}} \cong \mathrm{Hom}_{\mathbf{C}}(\mathbf{T}_{\mathbf{C}}, \mathbf{C}). \tag{12.2.1}$$

This isomorphism sends an element $T \in \mathbf{T}$ to the homomorphism

$$X \mapsto \langle v|T, X\rangle = a_1(v|T|X).$$

Since the identification $S_k = \mathrm{Hom}_{\mathbf{C}}(\mathbf{T}_{\mathbf{C}}, \mathbf{C})$ is canonical and since the vector v is canonical, we see that the isomorphism (12.2.1) is canonical.

Recall that M_k has as basis the set of products $E_4^a E_6^b$, where $4a + 6b = k$, and S_k is the subspace of forms where the constant coefficient of their q-expansion is 0. Thus there is a basis of S_k consisting of forms whose q-expansions have coefficients in $\mathbf{Q}$. Let $S_k(\mathbf{Z}) = S_k \cap \mathbf{Z}[[q]]$, be the submodule of S_k generated by cusp forms with Fourier coefficients in $\mathbf{Z}$, and note that $S_k(\mathbf{Z}) \otimes \mathbf{Q} \cong S_k(\mathbf{Q})$. Also, the explicit formula $(\sum a_n q^n)|T_p = \sum a_{np} q^n + p^{k-1} \sum a_n q^{np}$ implies that the Hecke algebra $\mathbf{T}$ preserves $S_k(\mathbf{Z})$.

Proposition 12.2.1. *The Fourier coefficients of each f_i are totally real algebraic integers.*

Proof. The coefficient $a_n(f_i)$ is the eigenvalue of T_n acting on f_i. As observed above, the Hecke operator T_n preserves $S_k(\mathbf{Z})$, so the matrix $[T_n]$ of T_n with respect to a basis for $S_k(\mathbf{Z})$ has integer entries. The eigenvalues of T_n are algebraic integers, since the characteristic polynomial of $[T_n]$ is monic and has integer coefficients.

The eigenvalues are real since the Hecke operators are self-adjoint with respect to the Petersson inner product. □

Remark 12.2.2. A *CM field* is a quadratic imaginary extension of a totally real field. For example, when $n > 2$, the field $\mathbf{Q}(\zeta_n)$ is a CM field, with totally real subfield $\mathbf{Q}(\zeta_n)^+ = \mathbf{Q}(\zeta_n + 1/\zeta_n)$. More generally, one shows that the eigenvalues of any newform $f \in S_k(\Gamma_1(N))$ generate a totally real or CM field.

Proposition 12.2.3. *We have $v \in S_k(\mathbf{Z})$.*

Proof. This is because $v = \sum \mathrm{Tr}(T_n) q^n$, and, as we observed above, there is a basis so that the matrices T_n have integer coefficients. □

Example 12.2.4. When $k = 36$, we have

$$\begin{aligned} v = 3q &+ 139656q^2 - 104875308q^3 + 34841262144q^4 + 892652054010q^5 \\ &- 4786530564384q^6 + 878422149346056q^7 + \cdots. \end{aligned}$$

The normalized newforms f_1, f_2, f_3 are

$$\begin{aligned} f_i = q + aq^2 &+ (-1/72a^2 + 2697a + 478011548)q^3 + (a^2 - 34359738368)q^4 \\ &(a^2 - 34359738368)q^4 + (-69/2a^2 + 14141780a + 1225308030462)q^5 + \cdots, \end{aligned}$$

for a each of the three roots of $X^3 - 139656X^2 - 59208339456X - 1467625047588864$.

12.2.2 General Level

Now we consider the case for general level N. Recall that there are maps

$$S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N)),$$

for all M dividing N and all divisor d of N/M.

The *old subspace* of $S_k(\Gamma_1(N))$ is the space generated by all images of these maps with $M|N$ but $M \neq N$. The *new subspace* is the orthogonal complement of the old subspace with respect to the Petersson inner product.

There is an algebraic definition of the new subspace. One defines trace maps

$$S_k(\Gamma_1(N)) \to S_k(\Gamma_1(M))$$

for all $M < N$, $M \mid N$ which are adjoint to the above maps (with respect to the Petersson inner product). Then f is in the new part of $S_k(\Gamma_1(N))$ if and only if f is in the kernels of all of the trace maps.

It follows from Atkin-Lehner-Li theory that the T_n acts semisimply on the new subspace $S_k(\Gamma_1(M))_{\text{new}}$ for all $M \geq 1$, since the common eigenspaces for all T_n each have dimension 1. Thus $S_k(\Gamma_1(M))_{\text{new}}$ has a basis of normalized eigenforms. We have a natural map

$$\bigoplus_{M|N} S_k(\Gamma_1(M))_{\text{new}} \hookrightarrow S_k(\Gamma_1(N)).$$

The image in $S_k(\Gamma_1(N))$ of an eigenform f for some $S_k(\Gamma_1(M))_{\text{new}}$ is called a *newform* of level $M_f = M$. Note that a newform of level less than N is not necessarily an eigenform for all of the Hecke operators acting on $S_k(\Gamma_1(N))$; in particular, it can fail to be an eigenform for the T_p, for $p \mid N$.

Let

$$v = \sum_f f(q^{\frac{N}{M_f}}) \in S_k(\Gamma_1(N)),$$

where the sum is taken over all newforms f of weight k and some level $M \mid N$. This generalizes the v constructed above when $N = 1$ and has many of the same good properties. For example, $S_k(\Gamma_1(N))$ is free of rank 1 over $\mathbf{T}$ with basis element v. Moreover, the coefficients of v lie in $\mathbf{Z}$, but to show this we need to know that $S_k(\Gamma_1(N))$ has a basis whose q-expansions lie in $\mathbf{Q}[[q]]$. This is true, but we will not prove it here. One way to proceed is to use the Tate curve to construct a q-expansion map $\mathrm{H}^0(X_1(N), \Omega_{X_1(N)/\mathbf{Q}}) \to \mathbf{Q}[[q]]$, which is compatible with the usual Fourier expansion map.

Example 12.2.5. The space $S_2(\Gamma_1(22))$ has dimension 6. There is a single newform of level 11,

$$f = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots .$$

There are four newforms of level 22, the four $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-conjugates of

$$g = q - \zeta q^2 + (-\zeta^3 + \zeta - 1)q^3 + \zeta^2 q^4 + (2\zeta^3 - 2)q^5 + (\zeta^3 - 2\zeta^2 + 2\,\zeta - 1)q^6 - 2\zeta^2 q^7 + \ldots$$

where ζ is a primitive 10th root of unity.

12.3 Decomposing the Anemic Hecke Algebra

We first observe that it make no difference whether or not we include the Diamond bracket operators in the Hecke algebra. Then we note that the **Q**-algebra generated by the Hecke operators of index coprime to the level is isomorphic to a product of fields corresponding to the Galois conjugacy classes of newforms.

Proposition 12.3.1. *The operators* $\langle d\rangle$ *on* $S_k(\Gamma_1(N))$ *lie in* $\mathbf{Z}[\ldots,T_n,\ldots]$.

Proof. It is enough to show $\langle p\rangle \in \mathbf{Z}[\ldots,T_n,\ldots]$ for primes p, since each $\langle d\rangle$ can be written in terms of the $\langle p\rangle$. Since $p \nmid N$, we have that

$$T_{p^2} = T_p^2 - \langle p\rangle p^{k-1},$$

so $\langle p\rangle p^{k-1} = T_p^2 - T_{p^2}$. By Dirichlet's theorem on primes in arithmetic progression [34, VIII.4], there is another prime q congruent to p mod N. Since p^{k-1} and q^{k-1} are relatively prime, there exist integers a and b such that $ap^{k-1}+bq^{k-1}=1$. Then

$$\langle p\rangle = \langle p\rangle(ap^{k-1}+bq^{k-1}) = a({T_p}^2 - T_{p^2}) + b({T_q}^2 - T_{q^2}) \in \mathbf{Z}[\ldots,T_n,\ldots].$$

□

Let S be a space of cusp forms, such as $S_k(\Gamma_1(N))$ or $S_k(\Gamma_1(N),\varepsilon)$. Let

$$f_1,\ldots,f_d \in S$$

be representatives for the Galois conjugacy classes of newforms in S of level N_{f_i} dividing N. For each i, let $K_i = \mathbf{Q}(\ldots,a_n(f_i),\ldots)$ be the field generated by the Fourier coefficients of f_i.

Definition 12.3.2 (Anemic Hecke Algebra). The *anemic Hecke algebra* is the subalgebra

$$\mathbf{T}_0 = \mathbf{Z}[\ldots,T_n,\ldots : (n,N)=1] \subset \mathbf{T}$$

of $\mathbf{T}$ obtained by adjoining to $\mathbf{Z}$ only those Hecke operators T_n with n relatively prime to N.

Proposition 12.3.3. *We have* $\mathbf{T}_0 \otimes \mathbf{Q} \cong \prod_{i=1}^d K_i$.

The map sends T_n to $(a_n(f_1),\ldots,a_n(f_d))$. The proposition can be proved using the discussion above and Atkin-Lehner-Li theory, but we will not give a proof here.

Example 12.3.4.
When $S = S_2(\Gamma_1(22))$, then $\mathbf{T}_0 \otimes \mathbf{Q} \cong \mathbf{Q} \times \mathbf{Q}(\zeta_{10})$ (see Example 12.2.5). When $S = S_2(\Gamma_0(37))$, then $\mathbf{T}_0 \otimes \mathbf{Q} \cong \mathbf{Q} \times \mathbf{Q}$.