## 5.7 Improper Integrals

Exam 2 Wed Mar 1: 7pm-7:50pm in ??
Today: 7.8 Improper Integrals
Monday – president's day holiday (and almost my bday)
Next — 11.1 sequences

**Example 5.7.1.** Make sense of $\int_0^\infty e^{-x} dx$. The integrals

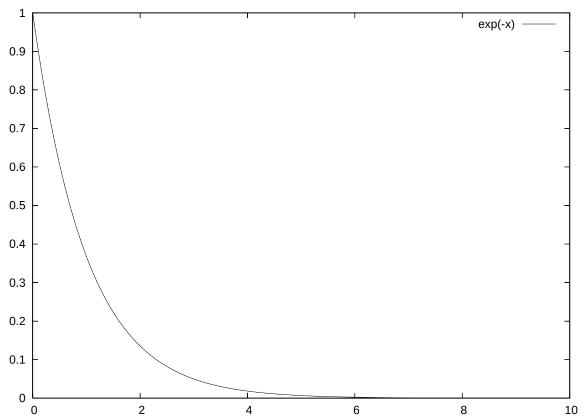
$$\int_0^t e^{-x} dx$$

make sense for each real number $t$. So consider

$$\lim_{t\to\infty} \int_0^t e^{-x} dx = \lim_{t\to\infty} [-e^{-x}]_0^t = 1.$$

Geometrically the area under the whole curve is the limit of the areas for finite values of $t$.

Figure 5.7.1: Graph of $e^{-x}$




**Example 5.7.2.** Consider $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ (see Figure 5.7.2). Problem: The denominator of the integrand tends to 0 as $x$ approaches the upper endpoint. Define

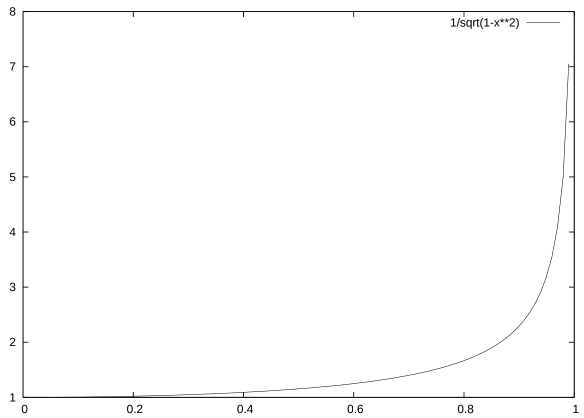
$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}} dx &= \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}} dx \\ &= \lim_{t\to 1^-} \left(\sin^{-1}(t) - \sin^{-1}(0)\right) = \sin^{-1}(1) = \frac{\pi}{2}\end{aligned}$$

Here $t \to 1^-$ means the limit as $t$ tends to 1 *from the left*.

**Example 5.7.3.** There can be multiple points at which the integral is improper. For example, consider

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

Figure 5.7.2: Graph of $\frac{1}{\sqrt{1-x^2}}$




A crucial point is that we take the limit for the left and right endpoints independently. We use the point 0 (for convenience only!) to break the integral in half.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \int_{-\infty}^{0} \frac{1}{1+x^2}dx + \int_{0}^{\infty} \frac{1}{1+x^2}dx \\ &= \lim_{s\to-\infty} \int_{s}^{0} \frac{1}{1+x^2}dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}dx \\ &= \lim_{s\to-\infty} (\tan^{-1}(0) - \tan^{-1}(s)) + \lim_{t\to\infty} (\tan^{-1}(t) - \tan^{-1}(0)) \\ &= \lim_{s\to-\infty} (-\tan^{-1}(s)) + \lim_{t\to\infty} (\tan^{-1}(t)) \\ &= -\frac{-\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

The graph of $\tan^{-1}(x)$ is in Figure 5.7.3.

**Example 5.7.4.** Brian Conrad's paper on impossibility theorems for elementary integration begins: "The Central Limit Theorem in probability theory assigns a special significance to the cumulative area function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2} udu$$

under the Gaussian bell curve

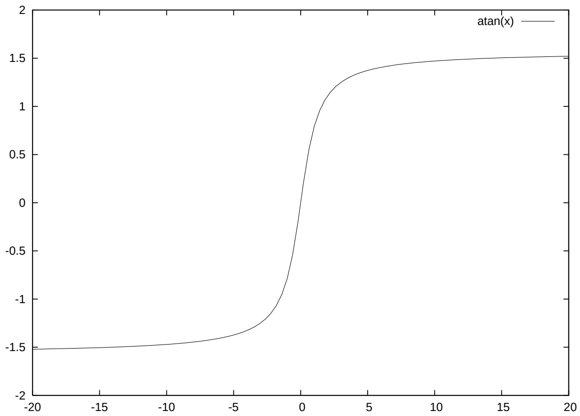
$$y = \frac{1}{\sqrt{2\pi}} \cdot e^{-u^2/2}.$$

It is known that $\Phi(\infty) = 1$."

What does this last statement *mean*? It means that

$$\lim_{t\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-t}^{0} e^{-u^2} udu + \lim_{x\to\infty} \frac{1}{\sqrt{2\pi}} \int_{0}^{x} e^{-u^2} udu = 1.$$

Figure 5.7.3: Graph of $\tan^{-1}(x)$




**Example 5.7.5.** Consider $\int_{-\infty}^{\infty} xdx$. Notice that

$$\int_{-\infty}^{\infty} xdx = \lim_{s\to-\infty}\int_s^0 xdx + \lim_{t\to\infty}\int_0^t xdx.$$

This diverges since each factor diverges independtly. But notice that

$$\lim_{t\to\infty}\int_{-t}^{t} xdx = 0.$$

This is *not* what $\int_{-\infty}^{\infty} xdx$ means (in this course – in a later course it could be interpreted this way)! This illustrates the importance of treating each bad point separately (since Example 5.7.3) doesn't.

**Example 5.7.6.** Consider $\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}dx$. We have

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}dx &= \lim_{s\to0^-}\int_{-1}^{s} x^{-\frac{1}{3}}dx \ + \ \lim_{t\to0^+}\int_{t}^{1} x^{-\frac{1}{3}}dx \\ &= \lim_{s\to0^-}\left(\frac{3}{2}s^{\frac{2}{3}} - \frac{3}{2}\right) + \lim_{t\to0^+}\left(\frac{3}{2} - \frac{3}{2}t^{\frac{2}{3}}\right) = 0.\end{aligned}$$

This illustrates how to be careful and break the function up into two pieces when there is a discontinuity.