3.4.1.

$$(x-2)(x-3)(x-2+1) = (x-2)(x-3)(x-1) = (x^2-5x+6)(x-1) = (x^3-6x^2+11x-6)$$

3.4.2. If P is a polynomial and $P(5) = 0$ then $(x-5)$ factors out. For example, **3.4.1** shows that $(x-1), (x-2), (x-3)$ factor out as they are roots. We'd have

$$\frac{(x^3-6x^2+11x-6)}{(x-1)} = \frac{(x^2-5x+6)(x-1)}{(x-1)} = (x^2-5x+6).$$

Hence $P(x) = (x-5)Q(x)$ for some other polynomial $Q(x)$ using the fundamental theorem of algebra. Above, $P(x)$ is played by $(x^3-6x^2+11x-6)$ and $Q(x)$ is played by (x^2-5x+6). Thus

$$\frac{P(x)}{(x-5)} = \frac{(x-5)Q(x)}{(x-5)} = Q(x).$$

3.4.3. To find poles, factor out the denominator as much as possible and check for poles on each factor.

(a) Note that $(5x^3+x+6) = (x+1)(5x^2-5x+6) = (x-(-1))(5x^2-5x+6)$. Therefore the only pole is at $x=-1$ as this is the only number such that $\lim_{x\to-1}|r(x)| = \infty$. There are no poles at ∞ as $\lim_{x\to\infty}|r(x)| \neq \infty$.

(b) The only pole is at $x = 2$ as $(x-2)(x^2+5x+7)$ is factored out as much as possible. There is no pole at infinity.

4.6.2.

$$\frac{2x+1}{(x-1)^2(x+2)} = \frac{Ax+B}{(x-1)^2} + \frac{C}{(x+2)}$$

To find A, B, C, I will (1) take limits on poles and (2) plug in values for x to solve the remaining variables. Note that

$$C = \lim_{x\to-2}(x+2)\frac{(2x+1)}{(x-1)^2(x+2)} = \frac{-3}{(-3)^2} = -\frac{1}{3}$$

To find B, set $x=0$ in the original equation and solve for B:

$$\frac{2(0)+1}{((0)-1)^2((0)+2)} = \frac{A(0)+B}{((0)-1)^2} + \frac{C}{((0)+2)} = B + \frac{C}{2}$$

Thus

$$\frac{1}{2} = B + \frac{C}{2} = B + \frac{-1/3}{2}$$

and solving for B gives

$$B = \frac{2}{3}.$$

To find A, set $x = 2$ in the original equation, substitite $B = 2/3, C = -1/3$ and solve for A. We get

$$\frac{2(2)+1}{(2-1)^2(2+2)} = \frac{2A+B}{(2-1)^2} + \frac{C}{(2+2)}$$

or

$$\frac{5}{4} = 2A + (2/3) + \frac{-1/3}{4}$$

so

$$A = \frac{1}{3}$$

Hence the decomposition is

$$\frac{2x+1}{(x-1)^2(x+2)} = \frac{(1/3)x + (2/3)}{(x-1)^2} + \frac{(-1/3)}{(x+2)}$$

4.6.3. We want

$$\lim_{x \to 2} \left| f(x) - \frac{A}{x-1} \right| = \infty, \qquad \lim_{x \to a} \left| f(x) - \frac{A}{x-1} \right| \neq \infty \text{ for } a \neq 2.$$

First combine terms to get

$$f(x) - \frac{A}{x-1} = \frac{-Ax^2 + 4Ax - 4A + 3}{(x-1)(x-2)^2}.$$

The only possible poles are $x = 1, x = 2$. This simplifies things as we only need to worry about making

$$\lim_{x \to 1} \left| f(x) - \frac{A}{x-1} \right| = \lim_{x \to 1} \left| \frac{-Ax^2 + 4Ax - 4A + 3}{(x-1)(x-2)^2} \right| \neq \infty.$$

The simplest way to do this is to choose A so that $-Ax^2 + 4Ax - 4A + 3$ has $(x-1)$ as a factor—which only happens if $-A(1)^2 + 4A(1) - 4A + 3 = 0$ by the fundamental theorem of algebra (or **3.4.2**). Solving we get $A = 3$.

4.6.4. To solve this problem, I will exclusively use poles with complex numbers (as this is the quickest and easiest method). I can do this because we don't have repeated factors [1].

Note that

$$(x^2 + 1) = (x+i)(x-i), (x^2 + 4) = (x+2i)(x-2i).$$

How did I get this? Remember **3.4.2**: if you plug in a certain number into a polynomial and get zero, that is a factor. For example—$(x^2 - 1)$ has $(1^2 - 1) = 0, ((-1)^2 - 1) = 0$. Therefore $(x^2 - 1) = (x-1)(x+1)$.

Same for $(x^2 + 1)$: we find $(i^2 + 1) = 0, ((-i)^2 + 1) = 0$—so $(x^2 + 1) = (x-i)(x+i)$. For a general quadratic $(ax^2 + bx + c)$ we get

$$ax^2 + bx + c = \left(x - \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \right) \left(x - \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) \right)$$

It looks really nasty but just plug in the numbers and it does the job. This comes from the quadratic formula—in short, the roots are the factors.

[1] For example: $\frac{x}{(x-1)^2(x+2)}$ has a repeated factor $(x-1)^2$ but $\frac{x}{(x-1)(x+2)}$ does not.

The (unsimplified) PFD is

$$\frac{x^3+2}{x(x^2+1)(x^2+4)} = \frac{A}{x} + \frac{B}{x-i} + \frac{C}{x+i} + \frac{D}{x-2i} + \frac{E}{x+2i}.$$

The coefficients are found as follows:

$$A = \lim_{x\to 0} x\frac{x^3+2}{x(x^2+1)(x^2+4)} = \frac{1}{2}$$

$$B = \lim_{x\to i}(x-i)\frac{x^3+2}{x(x^2+1)(x^2+4)} = \lim_{x\to i}(x-i)\frac{x^3+2}{x(x-i)(x+i)(x-2i)(x+2i)} = -\frac{2-i}{6}$$

$$C = \lim_{x\to -i}(x+i)\frac{x^3+2}{x(x^2+1)(x^2+4)} = \lim_{x\to -i}(x+i)\frac{x^3+2}{x(x-i)(x+i)(x-2i)(x+2i)} = -\frac{2+i}{6}$$

$$D = \lim_{x\to 2i}(x-2i)\frac{x^3+2}{x(x^2+1)(x^2+4)} = \lim_{x\to 2i}(x-2i)\frac{x^3+2}{x(x-i)(x+i)(x-2i)(x+2i)} = \frac{1-4i}{12}$$

$$E = \lim_{x\to -2i}(x+2i)\frac{x^3+2}{x(x^2+1)(x^2+4)} = \lim_{x\to -2i}(x+2i)\frac{x^3+2}{x(x-i)(x+i)(x-2i)(x+2i)} = \frac{1+4i}{12}$$

Hence

$$\frac{x^3+2}{x(x^2+1)(x^2+4)} = \frac{1/2}{x} + \frac{-(2-i)/6}{x-i} + \frac{-(2+i)/6}{x+i} + \frac{(1-4i)/12}{x-2i} + \frac{(1+4i)/12}{x+2i}$$

To simplify, I do as follows:

$$\frac{1/2}{x} + \frac{-(2-i)/6}{x-i}\left(\frac{x+i}{x+i}\right) + \frac{-(2+i)/6}{x+i}\left(\frac{x-i}{x-i}\right) + \frac{(1-4i)/12}{x-2i}\left(\frac{x+2i}{x+2i}\right) + \frac{(1+4i)/12}{x+2i}\left(\frac{x-2i}{x-2i}\right)$$

Multiplying out and combining everything (it takes some work, mind you), we get:

$$\frac{1/2}{x} - \frac{(2x+1)/3}{x^2+1} + \frac{(x+8)/6}{x^4+4}.$$

4.6.5. Same method as above, except we don't need to use complex numbers.

$$\frac{x^3+2}{x(x^2-1)(x^2-4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{D}{x-2} + \frac{E}{x+2}.$$

$$A = \lim_{x\to 0} x\frac{x^3+2}{x(x^2-1)(x^2-4)} = \frac{1}{2}$$

$$B = \lim_{x\to 1}(x-1)\frac{x^3+2}{x(x^2-1)(x^2-4)} = -\frac{1}{2}$$

$$C = \lim_{x \to -1} (x+1) \frac{x^3+2}{x(x^2-1)(x^2-4)} = -\frac{1}{6}$$

$$D = \lim_{x \to 2} (x-2) \frac{x^3+2}{x(x^2-1)(x^2-4)} = \frac{5}{12}$$

$$E = \lim_{x \to -2} (x+2) \frac{x^3+2}{x(x^2-1)(x^2-4)} = -\frac{1}{4}$$

Hence the PFD is:

$$\frac{x^3+2}{x(x^2-1)(x^2-4)} = \frac{1/2}{x} + \frac{-1/2}{x-1} + \frac{-1/6}{x+1} + \frac{5/12}{x-2} + \frac{-1/4}{x+2}.$$