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Project Summary

The Gross-Zagier Formula and Kolyvagin's Conjecture for Elliptic Curves of Higher Rank

In the 1960s, based on extensive numerical evidence, Birch and Swinnerton-Dyer conjectured that the algebraic and analytic ranks of any elliptic curve are equal, where the analytic rank is the order of vanishing of the associated Hasse-Weil L-function at 1. Their conjecture is proved for elliptic curves over the rational numbers when the analytic rank is at most 1, but little progress has been made when the rank is at least 2. The PI intends to explore three approaches to better understanding the conjecture when the rank is at least 2. The first approach involves a conjecture of Kolyvagin about Heegner points; the PI intends to verify the conjecture in specific cases for elliptic curves of rank at least 2 by explicitly computing cohomology classes, and prove results about how the cohomology classes are distributed. The second strategy involves the Gross-Zagier formula, where the PI intends to create new conjectural generalizations of the formula to higher rank, motivated by results and conjectures of Kolyvagin and others. The third strategy introduces elliptic curves over totally real fields; here the PI intends to compute tables, especially about elliptic curves of rank at least 2 and bounded conductor over totally real fields, generalize the other steps to totally real fields, and scrutinize cases in which the parameterizing Shimura curve has small genus.

Intellectual Merit: The proposed research could shed light on the Birch and Swinnerton-Dyer Conjecture, which is one of the central problems in number theory, e.g., it was chosen by the Clay Mathematics Institute as the Millennium Prize Problem in algebraic number theory. Explicit work in the 1960s and 1970s by Birch, Swinnerton-Dyer, Buhler, Stephens, Atkin, and others provided critical insight on which some of the great triumphs of Gross-Zagier, Kolyvagin, Wiles, and others in the 1980s and 1990s were based.

Broader Impact: The PI is co-authoring a popular expository book with Barry Mazur on the Riemann Hypothesis, co-authoring an advanced graduate level book with Kenneth Ribet on modular forms and Hecke operators, and intends to prepare a new edition of his AMS book on computing with modular forms. He has many tables of data that are freely available online, and whose creation has been supported by NSF FRG grant DMS-0757627, and the proposed research would expand these tables further. He will also continue to organize the development of the NSF-funded open source Sage mathematical software project that he started. The PI organizes dozens of workshops that involve many graduate students, that have a potentially broad impact on the mathematics community.