Notes for Algebraic Geometry II

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Contents

1 Preface

Read at your own risk!

These are my *very rough, error prone* notes of a second course on algebraic geometry offered at U.C. Berkeley in the Spring of 1996. The instructor was Robin Hartshorne and the students were Wayne Whitney, William Stein, Matt Baker, Janos Csirik, Nghi Nguyen, and Amod. I wish to thank Robin Hartshorne for giving this course and to Nghi Nguyen for his insightful suggestions and corrections. Of course all of the errors are solely my responsibility.

The remarks in brackets [[like this]] are notes that I wrote to myself. They are meant as a warning or as a reminder of something I should have checked but did not have time for. You may wish to view them as exercises.

If you have suggestions, questions, or comments feel free to write to me. My email address is `was@math.berkeley.edu`.

2 Ample Invertible Sheaves

Let k be an algebraically closed field and let X be a scheme over k. Let $\phi : X \to \mathbf{P}^n_k$ be a morphism. Then to give ϕ is equivalent to giving an invertible sheaf $\mathcal{L}$ on X and sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ which generate $\mathcal{L}$. If X is projective (that is, if there is some immersion of X into *some* $\mathbf{P}^m_k$) then ϕ is a closed immersion iff $s_0, \ldots, s_n$ separate points and tangent vectors.

Definition 2.1. Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Then we say $\mathcal{L}$ is *very ample* if there is an immersion $i : X \hookrightarrow \mathbf{P}^n_k$ such that $\mathcal{L} \cong i^*\mathcal{O}(1)$.

Theorem 2.2. *Let X be a closed subscheme of $\mathbf{P}^n_k$ and $\mathcal{F}$ a coherent sheaf on X, then $\mathcal{F}(n)$ is generated by global sections for all $n \gg 0$.*

Corollary 2.3. *Let X be any scheme and $\mathcal{L}$ a very ample coherent sheaf on X, then for all $n \gg 0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.*

Definition 2.4. Let X be a Noetherian scheme and $\mathcal{L}$ be an invertible sheaf. We say that $\mathcal{L}$ is *ample* if for every coherent sheaf $\mathcal{F}$ on X, there is n_0 such that for all $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections.

Thus the previous corollary says that a very ample invertible sheaf is ample.

Proposition 2.5. *Let X and $\mathcal{L}$ be as above. Then the following are equivalent.*
1) $\mathcal{L}$ is ample,
2) $\mathcal{L}^n$ is ample for all $n > 0$,
3) $\mathcal{L}^n$ is ample for some $n > 0$.

Theorem 2.6. *Let X be of finite type over a Noetherian ring A and suppose $\mathcal{L}$ is an invertible sheaf on A. Then $\mathcal{L}$ is ample iff there exists n such that $\mathcal{L}^n$ is very ample over* Spec A.

Example 2.7. Let $X = \mathbf{P}^1$, $\mathcal{L} = \mathcal{O}(\ell)$, some $\ell \in \mathbf{Z}$. If $\ell < 0$ then $\Gamma(\mathcal{L}) = 0$. If $\ell = 0$ then $\mathcal{L} = \mathcal{O}_X$ which is not ample since $\mathcal{O}_X(-1)^n \otimes \mathcal{O}_X \cong \mathcal{O}_X(-1)^n$ is not generated by global sections for any n. Note that $\mathcal{O}_X$ itself is generated by global sections. Finally, if $ell > 0$ then $\mathcal{L} = \mathcal{O}_X(\ell)$ is very ample hence ample.

Example 2.8. Let $C \subseteq \mathbf{P}^2$ be a nonsingular cubic curve and $\mathcal{L}$ an invertible sheaf on C defined by $\mathcal{L} = \mathcal{L}(D)$, where $D = \sum n_i P_i$ is a divisor on C. If $\deg D < 0$ then $\mathcal{L}$ has no global sections so it can't be ample.

3 Introduction to Cohomology

We first ask, what is cohomology and where does it arise in nature? Cohomology occurs in commutative algebra, for example in the Ext and Tor functors, it occurs in group theory, topology, differential geometry, and of course in algebraic geometry. There are several flavors of cohomology which are studied by algebraic geometers. Serre's coherent sheaf cohomology has the advantage of being easy to define, but has the property that the cohomology groups are vector spaces. Grothendieck introduced ètale cohomology and ℓ-adic cohomology. See, for example, Milne's *Ètale Cohomology* and SGA $4\frac{1}{2}$, 5 and 6. This cohomology theory arose from the study of the Weil Conjectures (1949) which deal with a deep relationship between the number of points on a variety over a finite field and the geometry of the complex analytic variety cut out by the same equations in complex projective space. Deligne was finally able to resolve these conjectures in the affirmative in 1974.

What is cohomology good for? Cohomology allows one to get numerical invariants of an algebraic variety. For example, if X is a projective scheme defined over an algebraically closed field k then $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space. Thus the $h_i = \dim_k H^i(X, \mathcal{F})$ are a set of numbers associated to X. "Numbers are useful in all branches of mathematics."

Example 3.1. Arithmetic Genus Let X be a nonsingular projective curve. Then $\dim H^1(X, \mathcal{O}_X)$ is the arithmetic genus of X. If $X \subseteq \mathbf{P}^n$ is a projective variety of dimension r then, if $p_a = \dim H^1(X, \mathcal{O}_X)$, then $1 + (-1)^r p_a =$ the constant term of the Hilbert polynomial of X.

Example 3.2. Let X be a nonsingular projective surface, then

$$1 + p_a = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_x)$$

and $1 + (-1)^r p_a = \chi(\mathcal{O}_X)$, the Euler characteristic of X.

Example 3.3. Let X be an algebraic variety and $\operatorname{Pic} X$ the group of Cartier divisors modular linear equivalence (which is isomorphic to the group of invertible sheaves under tensor product modulo isomorphism). Then $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*)$.

Example 3.4 ***(Deformation Theory).*** Let X_0 be a nonsingular projective variety. Then the first order infinitesimal deformations are classified by $H^1(X_0, T_{X_0})$ where T_{X_0} is the tangent bundle of X_0. The obstructions are classified by $H^2(X_0, T_{X_0})$.

One can define Cohen-Macaulay rings in terms of cohomology. Let $(A, \mathfrak{m})$ be a local Noetherian ring of dimension n, let $X = \operatorname{Spec} A$, and let $P = \mathfrak{m} \in X$, then we have the following.

Proposition 3.5. *Let A be as above. Then A is Cohen-Macaulay iff*
1) $H^0(X - P, \mathcal{O}_{X-P}) = A$ and
2) $H^i(X - P, \mathcal{O}_{X-P}) = 0$ for $0 < i < n - 1$.

A good place to get the necessary background for the cohomology we will study is in Appendices 3 and 4 from Eisenbud's *Commutative Algebra.*

4 Cohomology in Algebraic Geometry

For any scheme X and any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules we want to define the groups $H^i(X, \mathcal{F})$. We can either *define* cohomology by listing its properties, then later prove that we can construct the $H^i(X, \mathcal{F})$ or we can skip the definition and just construct the $H^i(X, \mathcal{F})$. The first method is more esthetically pleasing, but we will choose the second.

We first forget the scheme structure of X and regard X as a topological space and $\mathcal{F}$ as a sheaf of abelian groups (by ignoring the ring multiplication). Let $\mathbf{Ab}(X)$ be the category of sheaves of abelian groups on X. Let $\Gamma = \Gamma(X, \cdot)$ be the global section functor from $\mathbf{Ab}(X)$ into $\mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups. Recall that Γ is left exact so if

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence in $\mathbf{Ab}(X)$ then the following sequence is exact

$$0 \to \Gamma(\mathcal{F}') \to \Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'')$$

in $\mathbf{Ab}$.

Definition 4.1. We define the cohomology groups $H^i(X, \mathcal{F})$ to be the right derived functors of Γ.

5 Review of Derived Functors

The situation will often be as follows. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

a functor. Derived functors are the measure of the non-exactness of a functor. Let X be a topological space, $\mathbf{Ab}(X)$ the category of sheaves of abelian groups on X and $\mathbf{Ab}$ the category of abelian groups. Then $\Gamma(X, \cdot) : \mathbf{Ab}(X) \to \mathbf{Ab}$ is a left exact functor. Our cohomology theory will turn out to be the right derived function of $\Gamma(X, \cdot)$.

5.1 Examples of Abelian Categories

Although we will not define an abelian category we will give several examples and note that an abelian category is a category which has the same basic properties as these examples.

*Example 5.1 (A-**Modules**).* Let A be a fixed commutative ring and consider the category $\mathbf{Mod}(A)$ of A-modules. Then if M, N are any two modules one has

1) $\mathrm{Hom}(M, N)$ is an abelian group,
2) $\mathrm{Hom}(M, N) \times \mathrm{Hom}(N, L) \to \mathrm{Hom}(M, L)$ is a homomorphism of abelian groups.
3) there are kernels, cokernels, etc.

$\mathbf{Mod}(A)$ is an abelian category.

Example 5.2. Let A be a Noetherian ring and let our category be the collection of all finitely generated A-modules. Then this category is abelian. Note that if the condition that A be Noetherian is relaxed we may no longer have an abelian category because the kernel of a morphism of finitely generated modules over an arbitrary ring need not be finitely generated (for example, take the map from a ring to its quotient by an ideal which cannot be finitely generated).

Example 5.3. Let X be a topological space, then $\mathbf{Ab}(X)$ is an abelian category. If $(X, \mathcal{O}_X)$ is a ringed space then the category $\mathbf{Mod}(\mathcal{O}_X)$ is abelian. If X is a scheme then the category of quasi-coherent $\mathcal{O}_X$-modules is abelian, and if X is also Noetherian then the sub-category of coherent $\mathcal{O}_X$-modules is abelian.

Example 5.4. The category of abelian varieties is *not* an abelian category since the kernel of a morphism of abelian varieties might be reducible (for example an isogeny of degree n of elliptic curves has kernel n points which is reducible). It may be the case that the category of abelian group schemes is abelian but I don't know at the moment.

Example 5.5. The category of compact Hausdorff abelian topological groups is an abelian category.

5.2 Exactness

Definition 5.6. A functor $F : \mathcal{A} \to \mathcal{B}$ is *additive* if for all $X, Y \in \mathcal{A}$, the map

$$F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$$

is a homomorphism of abelian groups.

Definition 5.7. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* if $\mathrm{Im}(f) = \ker(g)$.

Definition 5.8. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor and

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence and consider the sequence

$$0 \to FM' \to FM \to FM'' \to 0.$$

If the second sequence is exact in the middle, then F is a called *half exact functor*. If the second sequence is exact on the left and the middle then F is called a *left exact functor*. If the second sequence is exact on the right and in the middle then we call F a *right exact functor*.

Example 5.9. Let A be a commutative ring and N an A-module. Then $N \otimes -$ is a right exact functor on the category of A-modules. To see that $N \otimes -$ is not exact, suppose $A, \mathfrak{m}$ is a local ring and $N = k = A/\mathfrak{m}$. Then the sequence

$$0 \to \mathfrak{m} \to A \to k \to 0$$

is exact, but

$$0 \to k \otimes \mathfrak{m} \to k \otimes A \to k \otimes k \to 0$$

is right exact but not exact.

Example 5.10. The functor $\mathrm{Tor}_1(N, \cdot)$ is neither left nor right exact.

Example 5.11. The contravarient hom functor, $\mathrm{Hom}(\cdot, N)$ is left exact.

Often the following is useful in work.

Theorem 5.12. *If*

$$0 \to M' \to M \to M''$$

is exact and F is left exact, then

$$0 \to FM' \to FM \to FM''$$

is exact.

5.3 Injective and Projective Objects

Let $\mathcal{A}$ be an abelian category. Then $\mathrm{Hom}_A(P, -) : \mathcal{A} \to \mathbf{Ab}$ is left exact.

Definition 5.13. An A module P is said to be *projective* if the functor $\mathrm{Hom}_A(P, -)$ is exact. An A module I is said to be *injective* if the functor $\mathrm{Hom}_A(-, I)$ is exact.

Definition 5.14. We say that an abelian category $\mathcal{A}$ has *enough projectives* if every X in $\mathcal{A}$ is the surjective image of a projective P in $\mathcal{A}$. A category is said to have *enough injectives* if every X in $\mathcal{A}$ injects into an injective objective of $\mathcal{A}$.

Example 5.15. Let A be a commutative ring, then $\mathbf{Mod}(A)$ has enough injectives because every module is the quotient of a free module and every free module is projective. If X is a topological space then $\mathbf{Ab}(X)$ has enough injectives. If X is a Noetherian scheme, then the category of quasi-coherent sheaves has enough injectives (hard theorem). The category of $\mathcal{O}_X$-modules has enough injectives but the category of coherent sheaves on X doesn't have enough injectives or projectives.

6 Derived Functors and Homological Algebra

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive covariant left-exact functor between abelian categories, for example $F = \Gamma : \mathbf{Ab}(X) \to \mathbf{Ab}$. Assume $\mathcal{A}$ has enough injectives, i.e., for all X in $\mathcal{A}$ there is an injective object I in $\mathcal{A}$ such that $0 \to X \hookrightarrow I$. We construct the right derived functors of F. If

$$0 \to M' \to M \to M'' \to 0$$

is exact in $\mathcal{A}$ then

$$0 \to F(M') \to F(M) \to F(M'') \to R^1F(M^1) \to R^1F(M) \to \cdots$$

is exact in $\mathcal{B}$ where R^iF is the right derived functor of F.

6.1 Construction of $R^i F$

Take any M in $\mathcal{A}$, then since $\mathcal{A}$ has enough injectives we can construct an exact sequence

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

where each I^i is an injective object. (This isn't totally obvious, but is a straightforward argument by putting together short exact sequences and composing maps.) The right part of the above sequence $I^0 \to I^1 \to \cdots$ is called an *injective resolution* of M. Applying F we get a complex

$$F(I^0) \xrightarrow{d_0} F(I^1) \xrightarrow{d_1} F(I^2) \xrightarrow{d_2} \cdots$$

in $\mathcal{B}$ which may not be exact. The objects $H^i = \ker(d_2)/\operatorname{Im}(d_1)$ measure the deviation of this sequence from being exact. H^i is called the ith *cohomology* object of the complex.

Definition 6.1. For each object in $\mathcal{A}$ fix an injective resolution. The ith *right derived functor* of F is the functor which assigns to an object M the ith cohomology of the complex $F(I^\cdot)$ where $I^\cdot$ is the injective resolution of M.

6.2 Properties of Derived Functors

We should now prove the following:

1. If we fix different injective resolutions for all of our objects then the corresponding derived functors are, in a suitable sense, isomorphic.

2. The $R^i F$ can also be defined on morphisms in such a way that they are really functors.

3. If $0 \to M' \to M \to M''$ is a short exact sequence then there is a long exact sequence of cohomology:

$$\begin{aligned} 0 \to FM' \to FM \to FM'' \to \\ R^1FM' \to R^1FM \to R^1FM'' \to \\ R^2FM' \to \cdots . \end{aligned}$$

4. If we have two short exact sequences then the induced maps on long exact sequences are "δ-compatible".

5. $R^0 F \cong F$.

6. If I is injective, then for any $i > 0$ one has that $R^i F(I) = 0$.

Theorem 6.2. *The $R^i F$ and etc. are uniquely determined by properties 1-6 above.*

Definition 6.3. A *δ-functor* is a collection of functors $\{R^i F\}$ which satisfy 3 and 4 above. An *augmented δ-functor* is a *δ-functor* along with a natural transformation $F \to R^0 F$. A *universal augmented δ-functor* is an *augmented δ-functor* with some universal property which I didn't quite catch.

Theorem 6.4. *If $\mathcal{A}$ has enough injectives then the collection of derived functors of F is a universal augmented δ-functor.*

To construct the $R^i F$ choose once and for all, for each object M in $\mathcal{A}$ an injective resolution, then prove the above properties hold.

7 Long Exact Sequence of Cohomology and Other Wonders

"Today I sat in awe as Hartshorne effortless drew hundreds of arrows and objects everywhere, chased some elements and proved that there is a long exact sequence of cohomology in 30 seconds. Then he whipped out his colored chalk and things really got crazy. Vojta tried to erase Hartshorne's diagrams during the next class but only partially succeeded joking that the functor was not 'effaceable'. (The diagrams are still not quite gone 4 days later!) Needless to say, I don't feel like texing diagrams and element chases... it's all trivial anyways, right?"

8 Basic Properties of Cohomology

Let X be a topological space, $\mathbf{Ab}(X)$ the category of sheaves of abelian groups on X and

$$\Gamma(X, \cdot) : \mathbf{Ab}(X) \to \mathbf{Ab}$$

the covariant, left exact global sections functor. Then we have constructed the derived functors $H^i(X, \cdot)$.

8.1 Cohomology of Schemes

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. To compute $H^i(X, \mathcal{F})$ forget all extra structure and use the above definitions. We may get some extra structure anyways.

Proposition 8.1. *Let X and $\mathcal{F}$ be as above, then the groups $H^i(X, \mathcal{F})$ are naturally modules over the ring $A = \Gamma(X, \mathcal{O}_X)$.*

Proof. Let $A = H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{O}_X)$ and let $a \in A$. Then because of the functoriality of $H^i(X, \cdot)$ the map $\mathcal{F} \to \mathcal{F}$ induced by left multiplication by a induces a homomorphism

$$a : H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}).$$

□

8.2 Objective

Our objective is to compute $H^i(\mathbf{P}^n_k, \mathcal{O}(\ell))$ for all i, n, ℓ. This is enough for most applications because if one knows these groups one can, in principle at least, computer the cohomology of any projective scheme. If X is any projective variety, we embed X in some $\mathbf{P}^n_k$ and push forward the sheaf $\mathcal{F}$ on X. Then we construct a resolution of $\mathcal{F}$ by sheaves of the form $\mathcal{O}(-\ell)^n$. Using Hilbert's syzigy theorem one sees that the resolution so constructed is finite and so we can put together our knowledge to get the cohomology of X.

Our plan of attack is as follows.

1. Define flasque sheaves which are acyclic for cohomology, i.e., the cohomology vanishes for $i > 0$.

2. If $X = \operatorname{Spec} A$, A Noetherian, and $\mathcal{F}$ is quasi-coherent, show that $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

3. If X is any Noetherian scheme and $\mathfrak{U} = (U_i)$ is an open affine cover, find a relationship between the cohomology of X and that of each U_i. (The "Čech process".)

4. Apply number 3 to $\mathbf{P}^n_k$ with $U_i = \{x_i \neq 0\}$.

9 Flasque Sheaves

Definition 9.1. A *flasque sheaf* (also called *flabby sheaf*) is a sheaf $\mathcal{F}$ on X such that whenever $V \subset U$ are open sets then $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Thus in a flasque sheaf, "every section extends".

Example 9.2. Let X be a topological space, $p \in X$ a point, not necessarily closed, and M an abelian group. Let $j : \{P\} \hookrightarrow X$ be the inclusion, then $\mathcal{F} = j_*(M)$ is flasque. This follows since

$$j_*(M)(U) = \begin{cases} M & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}.$$

Note that $j_*(M)$ is none other than the skyscraper sheaf at p with sections M.

Example 9.3. If $\mathcal{F}$ is a flasque sheaf on Y and $f : Y \to X$ is a morphism then $f_*\mathcal{F}$ is a flasque sheaf on X.

Example 9.4. If $\mathcal{F}_i$ are flasque then $\bigoplus_i \mathcal{F}_i$ is flasque.

Lemma 9.5. *If*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact and $\mathcal{F}'$ is flasque then

$$\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'') \to 0$$

is exact.

Lemma 9.6. *If*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact and $\mathcal{F}'$ and $\mathcal{F}$ are both flasque then $\mathcal{F}''$ is flasque.

Proof. Suppose $V \subset U$ are open subsets of X. Since $\mathcal{F}'$ is flasque and the restriction of a flasque sheaf is flasque and restriction is exact, lemma 1 implies that the sequence

$$\mathcal{F}(V) \to \mathcal{F}''(V) \to 0$$

is exact. We thus have a commuting diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}''(U) \arrow[d] & \\
\mathcal{F}(V) \arrow[r] & \mathcal{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

which, since $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective, implies $\mathcal{F}''(U) \to \mathcal{F}''(V)$ is surjective. $\square$

Lemma 9.7. *Injective sheaves (in the category of abelian sheaves) are flasque.*

Proof. Let $\mathcal{I}$ be an injective sheaf of abelian groups on X and let $V \subset U$ be open subsets. Let $s \in \mathcal{I}(V)$, then we must find $s' \in \mathcal{I}(U)$ which maps to s under the map $\mathcal{I}(U) \to \mathcal{I}(V)$. Let $\mathbf{Z}_V$ be the constant sheaf $\mathbf{Z}$ on V extended by 0 outside V (thus $\mathbf{Z}_V(W) = 0$ if $W \not\subset V$). Define a map $\mathbf{Z}_V \to \mathcal{I}$ by sending the section $1 \in \mathbf{Z}_V(V)$ to $s \in \mathcal{I}(V)$. Then since $\mathbf{Z}_V \hookrightarrow \mathbf{Z}_U$ and $\mathcal{I}$ is injective there is a map $\mathbf{Z}_U \to \mathcal{I}$ which sends the section $1 \in \mathbf{Z}_U$ to a section $s' \in \mathcal{I}(U)$ whose restriction to V must be s. $\square$

Remark 9.8. The same proof also shows that injective sheaves in the category of $\mathcal{O}_X$-modules are flasque.

Corollary 9.9. *If $\mathcal{F}$ is flasque then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Page 208 of [Hartshorne]. $\square$

Corollary 9.10. *Let $(X, \mathcal{O}_X)$ be a ringed space, then the derived functors of $\Gamma : \mathbf{Mod}\,\mathcal{O}_X \to \mathbf{Ab}$ are equal to $H^i(X, \mathcal{F})$.*

Proof. If

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

is an injective resolution of $\mathcal{F}$ in $\mathbf{Mod}\,\mathcal{O}_X$ then, by the above remark, it is a flasque resolution in the category $\mathbf{Ab}(X)$ hence we get the regular cohomology. $\square$

Remark 9.11. **Warning!** If $(X, \mathcal{O}_X)$ is a scheme and we choose an injective resolution in the category of quasi-coherent $\mathcal{O}_X$-modules then we are only guaranteed to get the right answer if X is Noetherian.

10 Examples

Example 10.1. Suppose C is a nonsingular projective curve over an algebraically closed field k. Let $K = K(C)$ be the function field of C and let $\mathcal{K}_C$ denote the constant sheaf K. Then we have an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{K}_C \to \bigoplus_{\substack{P \in C \\ P \text{ closed}}} K/\mathcal{O}_P \to 0,$$

where the map $\mathcal{K}_C \to \bigoplus K/\mathcal{O}_P$ has only finitely many components nonzero since a function $f \in K$ has only finitely many poles. Since C is irreducible $\mathcal{K}_C$ is flasque and since $\mathcal{K}/\mathcal{O}_P$ is a skyscraper sheaf it is flasque so since direct of flasque sheaves are flasque, $\bigoplus K/\mathcal{O}_P$ is flasque. One checks that the sequence is exact and so this is a flasque resolution of $\mathcal{O}_C$. Taking global sections and applying the exact sequence of cohomology gives an exact sequence

$$K \to \bigoplus_{P \text{ closed}} K/\mathcal{O}_P \to H^1(X, \mathcal{O}_C) \to 0,$$

and $H^i(X, \mathcal{O}_C) = 0$ for $i \geq 2$. Thus the only interesting information is $\dim_k H^1(X, \mathcal{O}_C)$ which is the *geometric genus* of C.

11 First Vanishing Theorem

> "Anyone who studies algebraic geometry must read French... looking up the more general version of this proof in EGA would be a good exercise."

Theorem 11.1. *Let A be a Noetherian ring, $X = \operatorname{Spec} A$ and $\mathcal{F}$ a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.*

Remark 11.2. The theorem is true without the Noetherian hypothesis on A, but the proof uses spectral sequences.

Remark 11.3. The assumption that $\mathcal{F}$ is quasi-coherent is essential. For example, let X be an affine algebraic curve over an infinite field k. Then X is homeomorphic as a topological space to $\mathbf{P}^1_k$ so the sheaf $\mathcal{O}(-2)$ on $\mathbf{P}^1_k$ induces a sheaf $\mathcal{F}$ of abelian groups on X such that

$$H^1(X, \mathcal{F}) \cong H^1(\mathbf{P}^1_k, \mathcal{O}(-2)) \neq 0.$$

Remark 11.4. If I is an injective A-module then $\tilde{I}$ need *not* be injective in $\mathbf{Mod}(\mathcal{O}_X)$ or $\mathbf{Ab}(X)$. For example, let $A = k = \mathbf{F}_p$ and $X = \operatorname{Spec} A$, then $I = k$ is an injective A-module but $\tilde{I}$ is the constant sheaf k. But k is a finite group hence not divisible so $\tilde{I}$ is not injective. (See Proposition A3.5 in Eisenbud's *Commutative Algebra.*)

Proposition 11.5. *Suppose A is Noetherian and I is an injective A-module, then $\tilde{I}$ is flasque on* $\operatorname{Spec} A$.

The proposition implies the theorem since if $\mathcal{F}$ is quasi-coherent then $\mathcal{F} = \tilde{M}$ for some A-module M. There is an injective resolution

$$0 \to M \to I^\bullet$$

which, upon applying the exact functor $\tilde{}$, gives a flasque resolution

$$0 \to \tilde{M} = \mathcal{F} \to \tilde{I}^\bullet.$$

Now applying Γ gives us back the original resolution

$$\Gamma : \quad 0 \to M \to I^\bullet$$

which is exact so the cohomology groups vanish for $i > 0$.

Proof. Let A be a Noetherian ring and I an injective A, then $\tilde{I}$ is a quasi-coherent sheaf on $X = \operatorname{Spec} A$. We must show that it is flasque. It is sufficient to show that for any open set U, $\Gamma(X) \to \Gamma(U)$ is surjective.

Case 1, special open affine: Suppose $U = X_f$ is a special open affine. Then we have a commutative diagram

$$\begin{array}{ccc} \Gamma(X, \tilde{I}) & \longrightarrow & \Gamma(X_f, \tilde{I}) \\ {\scriptstyle =}\big\downarrow & & {\scriptstyle =}\big\downarrow \\ I & \xrightarrow{\text{surjective?}} & I_f \end{array}$$

To see that the top map is surjective it is equivalent to show that $I \to I_f$ is surjective. This is a tricky algebraic lemma (see Hartshorne for proof).

Case 2, any open set: Let U be any open set. See Hartshorne for the rest.

∎

12 Čech Cohomology

Let X be a topological space, $\mathfrak{U} = (U_i)_{i \in I}$ an open cover and $\mathcal{F}$ a sheaf of abelian groups. We will define groups $\check{\mathrm{H}}^i(\mathfrak{U}, \mathcal{F})$ called Čech cohomology groups.

Warning: $\check{\mathrm{H}}^i(\mathfrak{U}, \cdot)$ is a functor in $\mathcal{F}$, but it is *not* a δ-functor.

Theorem 12.1. *Let X be a Noetherian scheme, $\mathfrak{U}$ an open cover and $\mathcal{F}$ a quasi-coherent sheaf, then $\check{H}^i(\mathfrak{U}, \mathcal{F}) = H^i(X, \mathcal{F})$ for all i.*

12.1 Construction

Totally order the index set I. Let

$$U_{i_0\cdots i_p} = \cap_{j=0}^p U_{i_j}.$$

For any $p \geq 0$ define

$$C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < i_1 < \cdots < i_p} \mathcal{F}(U_{i_0\cdots i_p}).$$

Then we get a complex

$$C^0(\mathfrak{U}, \mathcal{F}) \to C^1(\mathfrak{U}, \mathcal{F}) \to \cdots \to C^p(\mathfrak{U}, \mathcal{F}) \to \cdots$$

by defining a map

$$d : C^p(\mathfrak{U}, \mathcal{F}) \to C^{p+1}(\mathfrak{U}, \mathcal{F})$$

by, for $\alpha \in C^p(\mathfrak{U}, \mathcal{F})$,

$$(d\alpha)_{i_0\cdots i_{p+1}} := \sum_0^{p+1} (-1)^j \alpha_{i_0\cdots \hat{i_j} \cdots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}.$$

One checks that $d^2 = 0$.

Lemma 12.2. $\check{H}^0(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{F})$

Proof. Applying the sheaf axioms to the exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to C^0 = \prod_{i \in I} \mathcal{F}(U_i) \xrightarrow{d} C^1 = \prod_{i<j} \mathcal{F}(U_{ij})$$

we see that $\check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{F}) = \ker d = \Gamma(X, \mathcal{F})$. □

12.2 Sheafify

Let X be a topological space, $\mathfrak{U}$ an open cover and $\mathcal{F}$ a sheaf of abelian groups. Then we define

$$\mathcal{C}^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} j_*(\mathcal{F}|_{U_{i_0\cdots i_p}})$$

and define

$$d : \mathcal{C}^p(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathfrak{U}, \mathcal{F})$$

in terms of the d defined above by, for V an open set,

$$\mathcal{C}^p(\mathfrak{U}, \mathcal{F})(V) = C^p(\mathfrak{U}|_V, \mathcal{F}|_V) \xrightarrow{d} C^{p+1}(\mathfrak{U}|_V, \mathcal{F}|_V) = \mathcal{C}^{p+1}(\mathfrak{U}, \mathcal{F})(V).$$

Remark 12.3. $C^p(\mathfrak{U}, \mathcal{F}) = \Gamma(X, \mathcal{C}^p(\mathfrak{U}, \mathcal{F})$

Lemma 12.4. *The sequence*

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^1(\mathfrak{U}, \mathcal{F}) \to \cdots$$

is a resolution of $\mathcal{F}$*, i.e., it is exact.*

Proof. We define the map $\mathcal{F} \to \mathcal{C}^0$ by taking the product of the natural maps $\mathcal{F} \to f_*(\mathcal{F}|_{U_i})$, exactness then follows from the sheaf axioms.

To show the rest of the sequence is exact it suffices to show exactness at the stalks. So let $x \in X$, and suppose $x \in U_j$. Given $\alpha_x \in \mathcal{C}^p_x$ it is represented by a section $\alpha \in \Gamma(V, \mathcal{C}^p(\mathfrak{U}, \mathcal{F}))$, over a neighborhood V of x, which we may choose so small that $V \subset U_j$. Now for any p-tuple $i_0 < \ldots < i_{p-1}$, we set

$$(k\alpha)_{i_0,\ldots,i_{p-1}} = \alpha_{j,i_0,\ldots,i_{p-1}}.$$

This makes sense because

$$V \cap U_{i_0,\ldots,i_{p-1}} = V \cap U_{j,i_0,\ldots,i_{p-1}}.$$

Then take the stalk of $k\alpha$ at x to get the required map k.

Now we check that for any $p \geq 1$ and $\alpha \in \mathcal{C}^p_x$,

$$(dk + kd)(\alpha) = \alpha.$$

First note that

$$\begin{aligned}(dk\alpha)_{i_0,\ldots,i_p} &= \sum_{\ell=0}^{p} (-1)^\ell (k\alpha)_{i_0,\ldots,\hat{i_\ell},\ldots,i_p} \\ &= \sum (-1)^\ell \alpha_{j,i_0,\ldots,\hat{i_\ell},\ldots,i_p}\end{aligned}$$

Whereas, on the other hand,

$$\begin{aligned}(kd\alpha)_{i_0,\ldots,i_p} &= (d\alpha)_{j,i_0,\ldots,i_p} \\ &= (-1)^0 \alpha_{i_0,\ldots,i_p} + \sum_{\ell=1}^{p} (-1)^{\ell+1} \alpha_{j,i_0,\ldots,\hat{i_\ell},\ldots,i_p}\end{aligned}$$

Adding these two expressions yields $\alpha_{i_0,\ldots,i_p}$ as claimed.

Thus k is a homotopy operator for the complex $\mathcal{C}^\bullet_x$, showing that the identity map is homotopic to the zero map. It follows that the cohomology groups $H^p(\mathcal{C}^\bullet_x)$ of this complex are 0 for $p \geq 1$. $\square$

Lemma 12.5. *If $\mathcal{F}$ is flasque then $\mathcal{C}^p(\mathfrak{U}, \mathcal{F})$ is also flasque.*

Proof. If $\mathcal{F}$ is flasque then $\mathcal{F}|_{U_{i_0,\ldots,i_p}}$ is flasque so $j_*(\mathcal{F}|_{U_{i_0,\ldots,i_p}})$ is flasque so $\prod j_*(\mathcal{F}|_{U_{i_0,\ldots,i_p}})$ is flasque. $\square$

Proposition 12.6. *If $\mathcal{F}$ is flasque then $\check{H}^p(\mathfrak{U}, \mathcal{F}) = 0$.*

Proof. Consider the resolution

$$0 \to \mathcal{F} \to \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}).$$

By the above lemma it is flasque, so we can use it to compute the usual cohomology groups of $\mathcal{F}$. But $\mathcal{F}$ is flasque, so $H^p(X, \mathcal{F}) = 0$ for $p > 0$. On the other hand, the answer given by this resolution is

$$H^p(\Gamma(X, \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}))) = \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}).$$

So we conclude that $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = 0$ for $p > 0$. $\square$

Lemma 12.7. *Let X be a topological space, and $\mathfrak{U}$ an open covering. Then for each $p \geq 0$ there is a natural map, functorial in $\mathcal{F}$,*

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \to H^p(X, \mathcal{F}).$$

Theorem 12.8. *Let X be a Noetherian separated scheme, let $\mathfrak{U}$ be an open affine cover of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$ the natural maps give isomorphisms*

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \cong H^p(X, \mathcal{F}).$$

13 Čech Cohomology and Derived Functor Cohomology

Today we prove

Theorem 13.1. *Let X be a Noetherian, separated scheme, $\mathfrak{U}$ an open cover and $\mathcal{F}$ a quasi-coherent sheaf on X. Then*

$$\check{H}^i(\mathfrak{U}, \mathcal{F}) = H^i(X, \mathcal{F}).$$

To do this we introduce a condition (*):

Condition *: Let $\mathcal{F}$ be a sheaf of abelian groups and $\mathfrak{U} = (U_i)_{i \in I}$ an open cover. Then the pair $\mathcal{F}$ and $\mathfrak{U}$ satisfy condition (*) if for all $i_0, \dots, i_p \in I$,

$$H^(U_{i_0,\dots,i_p}, \mathcal{F}) = 0, \mathrm{all} i > 0.$$

Lemma 13.2. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence in $\mathbf{Ab}(X)$ and $\mathcal{F}'$ satisfies (*) then there is a long exact sequence for $\check{H}^i(\mathfrak{U}, \cdot)$.*

Proof. Since the global sections functor is left exact and cohomology commutes with products, we have an exact sequence

$$\begin{aligned} 0 \to C^p(\mathfrak{U}, \mathcal{F}') = \prod_{i_0 < \cdots < i_p} \mathcal{F}'(U_{i_0,\dots,i_p}) &\to C^p(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\dots,i_p}) \\ \to C^p(\mathfrak{U}, \mathcal{F}'') = \prod_{i_0 < \cdots < i_p} \mathcal{F}''(U_{i_0,\dots,i_p}) &\to \prod_{i_0 < \cdots < i_p} H^1(\mathfrak{U}_{i_0,\dots,i_p}, \mathcal{F}') = 0 \end{aligned}$$

where the last term is 0 because $\mathcal{F}'$ satisfies condition (*). Replacing p by $\cdot$ gives an exact sequence of complexes. Applying $\check{\mathrm{H}}^i(\mathfrak{U}, \cdot)$ then gives the desired result. □

Theorem 13.3. *Let X be a topological space, $\mathfrak{U}$ an open cover and $\mathcal{F} \in \mathbf{Ab}(X)$. Suppose $\mathcal{F}$ and $\mathfrak{U}$ satisfy (*). Then the maps*

$$\varphi^i : \check{H}^i(\mathfrak{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$$

are isomorphisms.

Proof. The proof is a clever induction. □

Lemma 13.4. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact and $\mathcal{F}'$ and $\mathcal{F}$ satisfy (*) then $\mathcal{F}''$ satisfies (*).*

To prove the main theorem of the section use the fact that X separated implies any finite intersection of affines is affine and then use the vanishing theorem for cohomology of a quasi-coherent sheaf on an affine scheme. The above theorem then implies the main result. From now on we will always assume our schemes are separated unless otherwise stated.

Corollary 13.5. *If X is a (separated) Noetherian scheme and X can be covered by $n+1$ open affines for some $n > 0$ then $H^i(X, \mathcal{F}) = 0$ for $i > n$.*

Example 13.6. Let $X = \mathbf{P}^n_k$, then the existence of the standard affine cover $U_0, \dots, U_n$ implies that $H^i(X, \mathcal{F}) = 0$ for $i > n$.

Example 13.7. Let X be a projective curve embedded in $\mathbf{P}^k_n$. Let $U_0 \subset X$ be open affine, then $X - U_0$ is finite. Thus $U_0 \subset X \subset \mathbf{P}^n$ and $X - U_0 = \{P_1, \dots, P_r\}$. In $\mathbf{P}^n$ there is a hyperplane H such that $P_1, \dots, P_r \notin H$. Then $P_1, \dots, P_r \in \mathbf{P}^n - H = \mathbf{A}^n = V$. Then $U_1 = V \cap X$ is closed in the affine set V, hence affine. Then $X = U_0 \cup U_1$ with U_0 and U_1 both affine. Thus $H^i(X, \mathcal{F}) = 0$ for all $i \geq 2$.

Exercise 13.8. If X is any projective scheme of dimension n then X can be covered by $n+1$ open affines so

$$H^i(X, \mathcal{F}) = 0 \text{ for all } i > n.$$

[Hint: Use induction.]

Hartshorne was unaware of the answer to the following question today.

Question 13.9. *If X is a Noetherian scheme of dimension n do there exist $n+1$ open affines covering X.*

Theorem 13.10 (Grothendieck). *If $\mathcal{F} \in \mathbf{Ab}(X)$ then $H^i(X, \mathcal{F}) = 0$ for all $i > n = \dim X$.*

Example 13.11. Let k be an algebraically closed field. Then $X = \mathbf{A}_k^2 - \{(0,0)\}$ is not affine since it has global sections $k[x,y]$. We compute $H^1(X, \mathcal{O}_X)$ by Čechcohomology. Write $X = U_1 \cup U_2$ where $U_1 = \{x \neq 0\}$ and $U_2 = \{y \neq 0\}$. Then the Čechcomplex is

$$C^{\cdot}(\mathfrak{U}, \mathcal{O}_X) : k[x,y,x^{-1}] \oplus k[x,y,y^{-1}] \xrightarrow{d} k[x,x^{-1},y,y^{-1}].$$

Thus one sees with a little thought that $H^0 = \ker d = k[x,y]$ and $H^1 = \{\sum_{i,j<0} a_{ij} x^i x^j : a_{ij} \in k\} = E$ as k-vector spaces (all sums are finite).

13.1 History of this Module E

$$E = \{\sum_{i,j<0} a_{ij} x^i x^j : a_{ij} \in k\}$$

1. Macaulay's "Inverse System" (1921?)
2. E is an injective A-module, in fact, the indecomposable injective associated to the prime (x,y)
3. E is the dualizing module of A, thus $D = \operatorname{Hom}_A(\cdot, E)$ is a dualizing functor for finite length modules (so doing D twice gives you back what you started with).
4. Local duality theorem: this is the module you "hom into".

14 Cohomology of $\mathbf{P}_k^n$

Today we begin to compute $H^i(X, \mathcal{O}_X(\ell))$ for all i and all ℓ.

a) $H^0(X, \mathcal{O}_X(\ell))$ is the vector space of forms of degree ℓ in $S = k[x_0, \ldots, x_n]$, thus

$$\oplus_{\ell \in \mathbf{Z}} H^0(\mathcal{O}_X(\ell)) = H^0_*(\mathcal{O}_X) = \Gamma_*(\mathcal{O}_X) = S.$$

Proposition 14.1. *There is a natural map*

$$H^0(\mathcal{O}_X(\ell)) \times H^i(\mathcal{O}_X(m)) \to H^i(\mathcal{O}_X(\ell + m)).$$

Proof. $\alpha \in H^0(\mathcal{O}_X(\ell))$ defines a map $\mathcal{O}_X \to \mathcal{O}_X(\ell)$ given by $1 \mapsto \alpha$. This defines a map

$$\mathcal{O}_X \otimes \mathcal{O}_X(m) \xrightarrow{\alpha(m)} \mathcal{O}_X(\ell) \otimes \mathcal{O}_X(m)$$

which gives a map $\mathcal{O}_X(m) \to \mathcal{O}_X(\ell + m)$. This induces the desired map $H^i(\mathcal{O}_X(m)) \to H^i(\mathcal{O}_X(\ell + m))$. $\square$

b) $H^i(\mathcal{O}_X(\ell)) = 0$ when $0 < i < n$ and for all ℓ. (This doesn't hold for arbitrary quasi-coherent sheaves!)

c) $H^n_*(X, \mathcal{O}_X)$ is a graded S-module which is 0 in degrees $\geq -n$, but is nonzero in degrees $\leq n-1$. As a k-vector space it is equal to

$$\{\sum_{i_j<0} a_{i_0,\dots,i_n} x_0^{i_0} \cdots x_n^{i_n} : \text{sum is finite}\}.$$

d) For $\ell \geq 0$ the map

$$H^0(\mathcal{O}_X(\ell)) \times H^n(\mathcal{O}_X(-\ell-n-1)) \to H^n(\mathcal{O}_X(-n-1)) \cong k$$

is a perfect pairing so we have a duality (which is in fact a special case of Serre Duality).

15 Serre's Finite Generation Theorem

We relax the hypothesis from the last lecture and claim that the same results are still true.

Theorem 15.1. *Let A be a Noetherian ring and $X = \mathbf{P}^n_A$. Then*

1. $H^0_*(\mathcal{O}_X) = \oplus_\ell H^0_\ell(\mathcal{O}_X(\ell)) = S = A[x_0, \dots, x_n]$
2. $H^i_*(\mathcal{O}_X) = 0$ *for all* $0 < i < n$
3. $H^n_*(\mathcal{O}_X) = \{\sum_{I=i_0,\dots,i_n} a_I x_0^{i_0} \cdots x_n^{i_n} : a_I \in A\}$
4. $H^0(\mathcal{O}_X(\ell)) \times H^n(\mathcal{O}_X(-\ell-n-1)) \to H^n(\mathcal{O}_X(-n-1))$ *is a perfect pairing of free A-modules. Notice that $H^n(\mathcal{O}_X(-n-1))$ is a free A-module of rank 1 so it is isomorphic to A, but* not *in a canonical way!*

Although pairing is in general not functorial as a map into A, there is a special situation in which it is. Let $\Omega^1_{X/k}$ be the sheaf of differentials on $X = \mathbf{P}^n_k$. Let $\omega = \Omega^n_{X/k} = \Lambda^n \Omega^1$ be the top level differentials (or "dualizing module"). Then some map is functorial (??)

"Is ω more important than Ω?" – Janos Csirik

"That's a value judgment... you can make your own decision on that... I won't."
– Hartshorne

Theorem 15.2 (Serre). *Let X be a projective scheme over a Noetherian ring A. Let $\mathcal{F}$ be any coherent sheaf on X. Then*

1. $H^i(X, \mathcal{F})$ *is a finitely generated A-module for all i*
2. *for all $\mathcal{F}$ there exists n_0 such that for all $i > 0$ and for all $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$.*

The following was difficult to prove last semester and we were only able to prove it under somewhat restrictive hypothesis on A (namely, that A is a finitely generated k-algebra).

Corollary 15.3. $\Gamma(X, \mathcal{F})$ *is a finitely generated A-module.*

Proof. Set $i = 0$ in 1. □

Proof. (of theorem)

I. *Reduce to the case $X = \mathbf{P}^r_A$.* Use the fact that the push forward of a closed subscheme has the same cohomology to replace $\mathcal{F}$ by $i_*(\mathcal{F})$.

II. *Special case, $\mathcal{F} = \mathcal{O}_{\mathbf{P}^r}(\ell)$ any $\ell \in \mathbf{Z}$.* 1. and 2. both follow immediately from the previous theorem. This is where we have done the work in explicit calculations.

III. *Cranking the Machine of Cohomology*

□

15.1 Application: The Arithmetic Genus

Let k be an algebraically closed field and $V \subset X = \mathbf{P}^n_k$ a projective variety. The arithmetic genus of V is

$$p_a = (-1)^{\dim V}(p_V(0) - 1)$$

where p_V is the Hilbert polynomial of V, thus $p_V(\ell) = \dim_k(S/I_V)_\ell$ for all $\ell \gg 0$. The Hilbert polynomial depends on the projective embedding of V.

Proposition 15.4. $p_V(\ell) = \sum_{i=0}^{\infty}(-1)^i \dim_k H^i(\mathcal{O}_V(\ell))$ *for* all $\ell \in \mathbf{Z}$.

This redefines the Hilbert polynomial. Furthermore,

$$p_a = (-1)^{\dim V}(p_V(0) - 1) = (-1)^{\dim V}\sum_{i=0}^{\infty}(-1)^i \dim_k(H^i(\mathcal{O}_V))$$

which shows that p_a is intrinsic, i.e., it doesn't depend on the embedding of V in projective space.

16 Euler Characteristic

Fix an algebraically closed field k, let $X = \mathbf{P}^n_k$. Suppose $\mathcal{F}$ is a coherent sheaf on X. Then by Serre's theorem $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space. Let

$$h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F}).$$

Definition 16.1. The **Euler characteristic** of $\mathcal{F}$ is

$$\chi(\mathcal{F}) = \sum_{i=0}^{n}(-1)^i h^i(X, \mathcal{F}).$$

Thus χ is a function $\mathbf{Coh}(X) \to \mathbf{Z}$.

Lemma 16.2. *If k is a field and*

$$0 \to V_1 \to V_2 \to \cdots \to V_N \to 0$$

is an exact sequence of finite dimensional vector spaces, then $\sum_{i=1}^{N}(-1)^i \dim V_i = 0$.

Proof. Since every short exact sequence of vector spaces splits, the statement is true when $N = 3$. If the statement is true for an exact sequence of length $N - 1$ then, applying it to the exact sequence

$$0 \to V_2/V_1 \to V_3 \to \cdots \to V_N \to 0,$$

shows that $\dim V_2/V_1 - \dim V_3 + \cdots \pm \dim V_n = 0$ from which the result follows. □

Lemma 16.3. *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is an exact sequence of coherent sheaves on* X*, then*

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

Proof. Apply the above lemma to the long exact sequence of cohomology taking into account that $H^n(\mathcal{F}'') = 0$ by Serre's vanishing theorem. □

More generally, any map χ from an abelian category to $\mathbf{Z}$ is called additive if, whenever

$$0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^n \to 0$$

is exact, then

$$\sum_{i=0}^{n} (-1)^i \chi(\mathcal{F}^i) = 0.$$

Question. Given an abelian category $\mathcal{A}$ find an abelian group A and a map $X : \mathcal{A} \to A$ such that every additive function $\chi : \mathcal{A} \to G$ factor through $\mathcal{A} \xrightarrow{X} A$. In the category of coherent sheaves the Grothendieck group solves this problem.

Let $X = \mathbf{P}^n_k$ and suppose $\mathcal{F}$ is a coherent sheaf on X. The Euler characteristic induces a map

$$\mathbf{Z} \to \mathbf{Z} : n \mapsto \chi(\mathcal{F}(n)).$$

Theorem 16.4. *There is a polynomial $p_{\mathcal{F}} \in \mathbf{Q}[z]$ such that $p_{\mathcal{F}}(n) = \chi(\mathcal{F}(n))$ for all $n \in \mathbf{Z}$.*

The polynomial $p_{\mathcal{F}}(n)$ is called the Hilbert polynomial of $\mathcal{F}$. Last semester we defined the Hilbert polynomial of a graded module M over the ring $S = k[x_0, \dots, x_n]$. Define $\varphi_M : \mathbf{Z} \to \mathbf{Z}$ by $\varphi_M(n) = \dim_k M_n$. Then we showed that there is a unique polynomial p_M such that $p_M(n) = \varphi_M(n)$ for all $n \gg 0$.

Proof. We induct on $\dim(\operatorname{supp} \mathcal{F})$. If $\dim(\operatorname{supp} \mathcal{F}) = 0$ then $\operatorname{supp} \mathcal{F}$ is a union of closed points so $\mathcal{F} = \oplus_{i=1}^k \mathcal{F}_{p_i}$. Since each $\mathcal{F}_{p_i}$ is a finite dimensional k-vector space and $\mathcal{O}_X(n)$ is locally free, there is a non-canonical isomorphism $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(n) \cong \mathcal{F}$. Thus

$$\chi_{\mathcal{F}}(n) = h^0(\mathcal{F}(n)) = h^0(\mathcal{F}) = \sum_{i=1}^{k} \dim_k \mathcal{F}_{p_i}$$

which is a constant function, hence a polynomial.

Next suppose $\dim(\operatorname{supp} \mathcal{F}) = s$. Let $x \in S_1 = H^0(\mathcal{O}_X(1))$ be such that the hyperplane $H := \{x = 0\}$ doesn't contain any irreducible component of $\operatorname{supp} \mathcal{F}$. Multiplication by x defines a map $\mathcal{O}_X(-1) \xrightarrow{x} \mathcal{O}_X$ which is an isomorphism outside of H. Tensoring with $\mathcal{F}$ gives a map $\mathcal{F}(-1) \to \mathcal{F}$. Let $\mathcal{R}$ be the kernel and $\mathcal{Q}$ be the cokernel, then there is an exact sequence

$$0 \to \mathcal{R} \to \mathcal{F}(-1) \xrightarrow{x} \mathcal{F} \to \mathcal{Q} \to 0.$$

Now $\operatorname{supp} \mathcal{R} \cup \operatorname{supp} \mathcal{Q} \subset \operatorname{supp} \mathcal{F} \cap H$ so $\dim(\operatorname{supp} \mathcal{R}) \leq \dim(\operatorname{supp} \mathcal{F}) \cap H < \dim(\operatorname{supp} \mathcal{F})$ and $\dim(\operatorname{supp} \mathcal{Q}) \leq \dim(\operatorname{supp} \mathcal{F}) \cap H < \dim(\operatorname{supp} \mathcal{F})$ so by our induction hypothesis $\chi(\mathcal{Q}(n))$ and $\chi(\mathcal{R}(n))$ are polynomials. Twisting the above exact sequence by n and applying χ yields

$$\chi(\mathcal{F}(n)) - \chi(\mathcal{F}(n-1)) = \chi(\mathbf{Q}(n)) - \chi(\mathcal{R}(n)) = P_{\mathcal{Q}}(n) - P_{\mathcal{R}}(N).$$

Thus the first difference function of $\chi(\mathcal{F}(n))$ is a polynomial so $\chi(\mathcal{F}(n))$ is a polynomial. $\square$

Example 16.5. Let $X = \mathbf{P}^1$ and $\mathcal{F} = \mathcal{O}_X$. Then $S = k[x_0, x_1]$, $M = S$ and $\dim S_n = n + 1$. Thus $p_M(z) = z + 1$ and $p_M(n) = \varphi(n)$ for $n \geq -1$. Computing the Hilbert polynomial in terms of the Euler characteristic gives

$$\chi(\mathcal{F}(n)) = h^0(\mathcal{O}_X(n)) - h^1(\mathcal{O}_X(n)) = \begin{cases} (n+1) - 0 & n \geq -1 \\ 0 - (-n-1) = n + 1 & n \leq -2 \end{cases}$$

Thus $p_{\mathcal{F}}(n) = n + 1$.

The higher cohomology corrects the failure of the Hilbert polynomial in lower degrees.

17 Correspondence between Analytic and Algebraic Cohomology

Homework. Chapter III, 4.8, 4.9, 5.6.

Look at Serre's 1956 paper *Geometrie Algebraique et Geometrie Analytique* (GAGA). "What are the prerequisites?" asks Janos. "French," answers Nghi. "Is there an English translation" asks the class. "Translation? ... What for? It's so beautiful in the French," retorts Hartshorne.

Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_{\mathbf{C}}$ with its Zariski topology. Then we can associate to $\mathcal{F}$ a sheaf $\mathcal{F}^{\text{an}}$ on $\mathbf{P}^n_{\mathbf{C}}$ with its analytic topology. $\mathcal{F}$ is locally a cokernel of a morphism of free sheaves so we can define $\mathcal{F}^{\text{an}}$ by defining $\mathcal{O}_X^{\text{an}}$. The map

$$\mathbf{Coh}(\mathbf{P}^n_{\mathbf{C}}) \xrightarrow{\text{an}} \mathbf{Coh}^{\text{an}}(\mathbf{P}^n_{\mathbf{C}})$$

is an equivalence of categories and

$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{\text{an}}, \mathcal{F}^{\text{an}})$$

for all i. If $X/\mathbf{C}$ is affine the corresponding object $X^{\text{an}}_{\mathbf{C}}$ is a Stein manifold.

18 Arithmetic Genus

Let $X \hookrightarrow \mathbf{P}^n_k$ be a projective variety with k algebraically closed and suppose $\mathcal{F}$ is a coherent sheaf on X. Then

$$\chi(\mathcal{F}) = \sum (-1)^i h^i(\mathcal{F})$$

is the Euler characteristic of $\mathcal{F}$,

$$P_{\mathcal{F}}(n) = \chi(\mathcal{F}(n))$$

gives the Hilbert polynomial of $\mathcal{F}$ on X, and

$$p_a(X) = (-1)^{\dim X}(P_{\mathcal{O}_X}(0) - 1)$$

is the arithmetic genus of X. The arithmetic genus is independent of the choice of embedding of X into $\mathbf{P}^n_k$.

If X is a curve then

$$1 - p_a(X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X)$$

. Thus if X is an integral projective curve then $h^0(\mathcal{O}_X) = 1$ so $p_a(X) = h^1(\mathcal{O}_X)$. If X is a nonsingular projective curve then $p_a(X) = h^1(\mathcal{O}_X)$ is called **the genus** of X.

Let V_1 and V_2 be varieties, thus they are projective integral schemes over an algebraically closed field k. Then V_1 and V_2 are **birationally equivalent** if and only if $K(V_1) \cong K(V_2)$ over k, where $K(V_i)$ is the function field of V_i. V is **rational** if V is bironational to $\mathbf{P}^n_k$ for some n. Since a rational map on a nonsingular projective curve always extends, two nonsingular projective curves are birational if and only if they are isomorphic. Thus for nonsingular projective curves the genus g is a birational invariant.

18.1 The Genus of Plane Curve of Degree d

Let $C \subset \mathbf{P}^2_k$ be a curve of degree d. Then C is a closed subscheme defined by a single homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d, thus

$$C = \mathrm{Proj}(S/(f)).$$

Some possibilities when $d = 3$ are:

- $f : Y^2 - X(X^2 - 1)$, a nonsingular elliptic curve
- $f : Y^2 - X^2(X - 1)$, a nodal cubic
- $f : Y^3$, a tripled x-axis
- $f : Y(X^2 + Y^2 - 1)$, the union of a circle and the x-axis

Now we compute $p_a(C)$. Let $I = (f)$ with $\deg f = d$. Then

$$1 - p_a = h_0(\mathcal{O}_C) - h_1(\mathcal{O}_C) + h_2(\mathcal{O}_C) = \chi(\mathcal{O}_C).$$

We have an exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_{\mathbf{P}^2} \to \mathcal{O}_C \to 0.$$

Now $\mathcal{I}_C \cong \mathcal{O}_{\mathbf{P}^2}(-d)$ since $\mathcal{O}_{\mathbf{P}^2}(-d)$ can be thought of as being generated by $1/f$ on $D_+(f)$ and by something else elsewhere, and then multiplication by f gives an inclusion $so_{\mathbf{P}^2}(-d)|_{D_+(f)} \to \mathcal{O}_{\mathbf{P}^2}|_{D_+(f)}$, etc. Therefore

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_{\mathbf{P}^2}) - \chi(\mathcal{O}_{\mathbf{P}^2}(-d)).$$

Now

$$\chi(\mathcal{O}_{\mathbf{P}^2}) = h^0(\mathcal{O}_{\mathbf{P}^2}) - h^1(\mathcal{O}_{\mathbf{P}^2}) + h^2(\mathcal{O}_{\mathbf{P}^2}) = 1 + 0 + 0$$

and

$$\chi(\mathcal{O}_{\mathbf{P}^2}(-d)) = h^0(\mathcal{O}_{\mathbf{P}^2}(-d)) - h^1(\mathcal{O}_{\mathbf{P}^2}(-d)) + h^2(\mathcal{O}_{\mathbf{P}^2}(-d)) = 0 + 0 + \frac{1}{2}(d-1)(d-2).$$

For the last computation we used duality (14.1) to see that

$$h^2(\mathcal{O}_{\mathbf{P}^2}(-d)) = h^0(\mathcal{O}_{\mathbf{P}^2}(d-3) = \dim S_{d-3} = \frac{1}{2}(d-1)(d-2).$$

Thus $\chi(\mathcal{O}_C) = 1 - \frac{1}{2}(d-1)(d-2)$ so

$$p_a(C) = \frac{1}{2}(d-1)(d-2).$$

19 Not Enough Projectives

Exercise 19.1. Prove that the category of quasi-coherent sheaves on $X = \mathbf{P}^1_k$ doesn't have enough projectives.

Proof. We show that there is no projective object $\mathcal{P} \in \mathbf{Qco}(X)$ along with a surjection $\mathcal{P} \to \mathcal{O}_X \to 0$.

Lemma 19.2. *If* $\mathbf{P} \xrightarrow{\varphi} \mathcal{O}_X$ *is surjective and* $\mathcal{P}$ *is quasi-coherent, then there exists* ℓ *such that* $H^0(\mathbf{P}(\ell)) \to H^0(\mathcal{O}_X(\ell))$ *is surjective.*

The *false* proof of this lemma is to write down an exact sequence $0 \to \mathcal{R} \to \mathcal{P} \to \mathcal{O}_X \to 0$ then use the "fact" that $H^1(\mathcal{R}(\ell)) = 0$ for sufficiently large ℓ. This doesn't work because $\mathcal{R}$ might not be coherent since it is only the quotient of quasi-coherent sheaves. A valid way to proceed is to use (II, Ex. 5.15) to write $\mathcal{P}$ as an ascending union of its coherent subsheaves, $\mathcal{P} = \cup_i \mathcal{P}_i$. Then since φ is surjective, $\mathcal{O}_X = \cup_i \varphi(\mathcal{P}_i)$, where $\varphi(\mathcal{P}_i)$ is the sheaf image. Using the fact that $\varphi(\mathcal{P}_i)$ is the sheaf image, that $\mathcal{O}_X$ is coherent and that the union is ascending, this implies $\mathcal{O}_X = \varphi(\mathcal{P}_i)$ for some i. We now have an exact sequence

$$0 \to \mathcal{R}_i \to \mathcal{P}_i \to \mathcal{O}_X \to 0$$

with $\mathcal{R}_i$ coherent since $\mathcal{P}_i$ and $\mathcal{O}_X$ are both coherent. Thus $H^i(\mathcal{R}_i(\ell)) = 0$ for $l \gg 0$ which, upon computing the long exact sequence of cohomology, gives the lemma.

Now fix such an ℓ. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[d, "\exists"'] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
\mathcal{O}_X(-\ell-1) \arrow[r] & k(p) \arrow[r] & 0
\end{tikzcd}$$

Twisting by ℓ gives a commutative diagram

$$\begin{tikzcd}
\mathcal{P}(\ell) \arrow[r] \arrow[d] & \mathcal{O}_X(\ell) \arrow[r] \arrow[d] & 0 \\
\mathcal{O}_X(-1) \arrow[r] & k(p) \arrow[r] & 0
\end{tikzcd}$$

Let $s \in \Gamma(\mathcal{O}_X(\ell))$ be a global section which is nonzero at p, then there is $t \in \Gamma(\mathcal{P}(\ell))$ which maps to s. But then by commutativity t must map to some element of $\Gamma(\mathcal{O}_X(-1)) = 0$ which maps to a nonzero element of $k(p)$, which is absurd. $\square$

20 Some Special Cases of Serre Duality

20.1 Example: $\mathcal{O}_X$ on Projective Space

Suppose $X = \mathbf{P}^n_k$, then there is a perfect pairing

$$H^0(\mathcal{O}_X(\ell)) \times H^n(\mathcal{O}_X(-\ell-n-1)) \to H^n(\mathcal{O}_X(-n-1)) \cong k.$$

For this section let

$$\omega_X = \mathcal{O}_X(-n-1).$$

Because the pairing is perfect we have a non-canonical but functorial isomorphism

$$H^0(\mathcal{O}_X(\ell)) \cong H^n(\mathcal{O}_X(-\ell-n-1))'.$$

(If V is a vector space then V' denotes its dual.)

20.2 Example: Coherent sheaf on Projective Space

Suppose $\mathcal{F}$ is any coherent sheaf on $X = \mathbf{P}^r_k$. View $\mathrm{Hom}(\mathcal{F}, \omega)$ as a k-vector space.

By functoriality and since $H^n(\omega) = k$ there is a map

$$\varphi : \mathrm{Hom}(\mathcal{F}, \omega) \to \mathrm{Hom}(H^n(\mathcal{F}), H^n(\omega)) = H^n(\mathcal{F})'.$$

Proposition 20.1. *φ is an isomorphism for all coherent sheaves $\mathcal{F}$.*

Proof. Case 1. If $\mathcal{F} = \mathcal{O}_X(\ell)$ for some $\ell \in \mathbf{Z}$ then this is just a restatement of the previous example.

Case 2. If $\mathcal{E} = \oplus_{i=1}^k \mathcal{O}(\ell_i)$ is a finite direct sum, then the statement follows from the commutativity of the following diagram.

$$\begin{array}{ccc}
\mathrm{Hom}(\oplus_{i=1}^k \mathcal{O}(\ell_i), \omega) & \longrightarrow & H^n(\oplus_{i=1}^k \mathcal{O}(\ell_i))' \\
\downarrow \cong & & \downarrow \cong \\
\oplus_{i=1}^k \mathrm{Hom}(\mathcal{O}(\ell_i), \omega) & \xrightarrow[\sim]{} & \oplus_{i=1}^k H^n(\mathcal{O}(\ell_i))'
\end{array}$$

Case 3. Now let $\mathcal{F}$ be an arbitrary coherent sheaf. View φ as a morphism of functors

$$\mathrm{Hom}(\cdot, \omega) \to H^n(\cdot)'.$$

The functor $\mathrm{Hom}(\cdot, \omega)$ is contravarient left exact. $H^n(\cdot)$ is covariant right exact since $X = \mathbf{P}^n_k$ so $H^{n+1}(\mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$. Thus $H^n(\cdot)'$ is contravarient left exact.

Lemma 20.2. *Let $\mathcal{F}$ be any coherent sheaf. Then there exists a partial resolution*

$$\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

by sheaves of the form $\oplus_i \mathcal{O}_X(\ell_i)$.

By (II, 5.17) for $\ell \gg 0$, $\mathcal{F}(\ell)$ is generated by its global sections. Thus there is a surjection

$$\mathcal{O}_X^m \to \mathcal{F}(\ell) \to 0$$

which upon twisting by $-\ell$ becomes

$$\mathcal{E}_0 = \mathcal{O}_X(-\ell)^m \to \mathcal{F} \to 0.$$

Let $\mathcal{R}$ be the kernel so

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F}$$

is exact. Since $\mathcal{R}$ is coherent, we can repeat the argument above to find $\mathcal{E}_1$ surjecting onto $\mathcal{R}$. This yields the desired exact sequence.

Now we apply the functors $\mathrm{Hom}(\cdot, \omega)$ and $H^n(\cdot)'$. This results in a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}, \omega) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_0, \omega) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_1, \omega) \\
\downarrow & & \varphi(\mathcal{F}) \downarrow & & \varphi(\mathcal{E}_0) \downarrow & & \downarrow \varphi(\mathcal{E}_1) \\
0 & \longrightarrow & H^n(\mathcal{F})' & \longrightarrow & H^n(\mathcal{E}_0)' & \longrightarrow & H^n(\mathcal{E}_1)'
\end{array}$$

From cases 1 and 2, the maps $\varphi(\mathcal{E}_0)$ and $\varphi(\mathcal{E}_1)$ are isomorphisms so $\varphi(\mathcal{F})$ must also be an isomorphism. $\square$

20.3 Example: Serre Duality on $\mathbf{P}_k^n$

Let $X = \mathbf{P}_k^n$ and $\mathcal{F}$ be a coherent sheaf. Then for each i there is an isomorphism

$$\varphi^i : \operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega) \to H^{n-i}(\mathcal{F})'.$$

21 The Functor Ext

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}(\mathcal{O}_X)$. Then $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \in \mathbf{Ab}$. View $\operatorname{Hom}(\mathcal{F}, \bullet)$ as a functor $\mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Ab}$. Note that $\operatorname{Hom}(\mathcal{F}, \bullet)$ is left exact and covariant. Since $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives we can take derived functors.

Definition 21.1. The Ext functors $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \bullet)$ are the right derived functors of $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \bullet)$ in the category $\mathbf{Mod}(\mathcal{O}_X)$.

Thus to compute $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, take an injective resolution

$$0 \to \mathcal{G} \to I^0 \to I^1 \to \cdots$$

then

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^i(\operatorname{Hom}_{\mathcal{O}_X(\mathcal{F}, I^\bullet)}).$$

Remark 21.2. **Warning!** If $i : X \hookrightarrow \mathbf{P}^n$ is a closed subscheme of P^n then $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ need *not* equal $\operatorname{Ext}^i_{\mathbf{P}^n}(i_*(\mathcal{F}), i_*(\mathcal{G}))$. With cohomology these are the same, but not with Ext!

Example 21.3. Suppose $\mathcal{F} = \mathcal{O}_X$, then $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = \Gamma(X, \mathcal{G})$. Thus $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \bullet)$ are the derived functors of $\Gamma(X, \bullet)$ in $\mathbf{Mod}(\mathcal{O}_X)$. Since we can computer cohomology using flasque sheaves this implies $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \bullet) = H^i(X, \bullet)$. Thus Ext generalizes H^i but we get a lot more besides.

21.1 Sheaf Ext

Now we define a new kind of Ext. The sheaf hom functor

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, \bullet) : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(\mathcal{O}_X)$$

is covariant and left exact. Since $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives we can defined the derived functors $\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F}, \bullet)$.

Example 21.4. Consider the functor $\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \bullet)$. Since $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$ this is the identity functor which is exact so

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{G}) = \begin{cases} \mathcal{G} & i = 0 \\ 0 & i > 0 \end{cases}$$

What if we have a short exact sequence in the first variables, do we get a long exact sequence?

Proposition 21.5. *The functors* Ext^i *and* $\mathcal{E}\mathrm{xt}^i$ *are* δ*-functors in the first variable. Thus if*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is exact then there is a long exact sequence

$$0 \to \operatorname{Hom}(\mathcal{F}'', \mathcal{G}) \to \operatorname{Hom}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}(\mathcal{F}', \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{F}'', \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{G}) \to \operatorname{Ext}^1(\mathcal{F}, \mathcal{G}) \to \cdots$$

The conclusion of this proposition is not obvious because we the Ext^i as derived functors in the second variable, not the first.

Proof. Suppose we are given $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ and $\mathcal{G}$. Choose an injective resolution $0 \to \mathcal{G} \to I^\bullet$ of $\mathcal{G}$. Since $\mathrm{Hom}(\bullet, I^n)$ is exact (by definition of injective object), the sequence

$$0 \to \mathrm{Hom}(\mathcal{F}'', I^\bullet) \to \mathrm{Hom}(\mathcal{F}, I^\bullet) \to \mathrm{Hom}(\mathcal{F}', I^\bullet) \to 0$$

is exact. By general homological algebra these give rise a long exact sequence of cohomology of these complexes. For $\mathcal{E}\mathrm{xt}^i$ simply scriptify everything! □

21.2 Locally Free Sheaves

Proposition 21.6. *Suppose $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of finite rank. Let $\mathcal{E}^\vee = \mathrm{Hom}(\mathcal{E}, \mathcal{O})$. For any sheaves $\mathcal{F}$, $\mathcal{G}$,*

$$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}^\vee)$$

and

$$\mathcal{E}\mathrm{xt}^i(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \mathcal{E}\mathrm{xt}^i(\mathcal{F}, \mathcal{G} \otimes \mathcal{E}^\vee) \cong \mathcal{E}\mathrm{xt}^i(\mathcal{F}, \mathcal{G}) \otimes \mathcal{E}^\vee.$$

Lemma 21.7. *If $\mathcal{E}$ is locally free of finite rank and $\mathcal{I} \in \mathbf{Mod}(\mathcal{O}_X)$ is injective then $\mathcal{I} \otimes \mathcal{E}$ is injective.*

Proof. Suppose $0 \to \mathcal{F} \to \mathcal{G}$ is an injection and there is a map $\varphi : \mathcal{F} \to \mathcal{I} \otimes \mathcal{E}$. Tensor everything with $\mathcal{E}^\vee$. Then we have an injection $0 \to \mathcal{F} \otimes \mathcal{E}^\vee \to \mathcal{G} \otimes \mathcal{E}^\vee$ and a map $\varphi' : \mathcal{F} \otimes \mathcal{E}^\vee \to \mathcal{I}$. Since $\mathcal{I}$ is injective there is a map $\mathcal{G} \otimes \mathcal{E}^\vee \to \mathcal{I}$ which makes the appropriate diagram commute. Tensoring everything with $\mathcal{E}$ gives a map making the original diagram commute. □

of proposition. Let $0 \to \mathcal{G} \to \mathcal{I}^\bullet$ by an injective resolution of $\mathcal{G}$. Since

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{E}, \mathcal{I}^\bullet) = \mathrm{Hom}(\mathcal{F}, \mathcal{I}^\bullet \otimes \mathcal{E}^\vee),$$

we see that

$$0 \to \mathcal{G} \otimes \mathcal{E}^\vee \to \mathcal{I}^\cdot \otimes \mathcal{E}\vee$$

is an injective resolution of $\mathcal{G} \otimes \mathcal{E}^\vee$. Thus $\mathrm{Hom}(\mathcal{F}, \mathcal{I}^\bullet \otimes \mathcal{E}^\vee)$ computes $\mathrm{Ext}(\mathcal{F} \otimes \mathcal{E}, \bullet)$. □

Proposition 21.8. *If $\mathcal{F}$ has a locally free resolution $\mathcal{E}. \to \mathcal{F} \to 0$ then*

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H^i(\mathcal{H}\mathrm{om}(\mathcal{E}_\bullet, \mathcal{G})).$$

Remark 21.9. Notice that when $i > 0$ and $\mathcal{E}$ is locally free,

$$\mathcal{E}\mathrm{xt}^i(\mathcal{E}, \mathcal{G}) = \mathcal{E}\mathrm{xt}^i(\mathcal{O}_X, \mathcal{G} \otimes \mathcal{E}^\vee) = 0.$$

Proof. Regard both sides as functors in $\mathcal{G}$. The left hand side is a δ-functor and vanishes for $\mathcal{G}$ injective. I claim that that right hand side is also a δ-functor and vanishes for $\mathcal{G}$-injective.

Lemma 21.10. *If $\mathcal{E}$ is locally free, then $\mathcal{H}\mathrm{om}(\mathcal{E}, \bullet)$ is exact.*

□

22 More Technical Results on Ext

Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}, \mathcal{G}$ be sheaves in the category $\mathbf{Mod}(\mathcal{O}_X)$. Then $\mathrm{Ext}(\mathcal{F}, \mathcal{G})$ and $\mathcal{E}\mathrm{xt}(\mathcal{F}, \mathcal{G})$ are the derived functors of Hom, resp. $\mathcal{H}\mathrm{om}$, in the second variable.

Lemma 22.1. *If $\mathcal{F}$ and $\mathcal{G}$ are coherent over a Noetherian scheme X, then $\mathcal{E}\mathrm{xt}^i(\mathcal{F}, \mathcal{G})$ is coherent.*

This lemma would follow immediately from the following fact which we haven't proved yet.

Fact 22.2. *Let $X = \operatorname{Spec} A$ with A Noetherian and let M be an A-module. Then*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) = \mathrm{Ext}^i_A(M, N)$$

and

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\tilde{M}, \tilde{N}) = (\mathrm{Ext}^i_A(M, N))\tilde{}.$$

Instead of using the fact we can prove the lemma using a proposition from yesterday.

Proof. Choose a locally free resolution

$$\mathcal{L}_\bullet \to \mathcal{F} \to 0$$

of $\mathcal{F}$. Then

$$\mathcal{E}\mathrm{xt}^i(\mathcal{F}, \mathcal{G}) = H^i(\mathcal{H}\mathrm{om}(\mathcal{L}_\bullet, \mathcal{G})).$$

But all of the kernels and cokernels in $\mathcal{H}\mathrm{om}(\mathcal{L}_\bullet, \mathcal{G})$ are coherent, so the cohomology is. (We can't just choose an injective resolution of $\mathcal{F}$ and apply the definitions because there is no guarantee that we can find an injective resolution by coherent sheaves.) $\square$

Proposition 22.3. *Let X be a Noetherian projective scheme over k and let $\mathcal{F}$ and $\mathcal{G}$ be coherent on X. Then for each i there exists an n_0, depending on i, such that for all $n \geq n_0$,*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n)) = \Gamma(\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}(n))).$$

Proof. When $i = 0$ the assertion is that

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}(n)) = \Gamma(\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G}(n)))$$

which is obvious.

Claim. Both sides are δ-functors in $\mathcal{F}$. We have already showed this for the left hand side. [I don't understand why the right hand side is, but it is not trivial and it caused much consternation with the audience.]

To show the functors are isomorphic we just need to show both sides are coeffaceable. That is, for every coherent sheaf $\mathcal{F}$ there is a coherent sheaf $\mathcal{E}$ and a surjection $\mathcal{E} \to \mathcal{F} \to 0$ such that $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{E}) = 0$ and similarly for the right hand side. Thus every coherent sheaf is a quotient of an acyclic sheaf.

Suppose $\mathcal{F}$ is coherent. Then for $\ell \gg 0$, $\mathcal{F}(\ell)$ is generated by its global sections, so there is a surjection

$$\mathcal{O}_X^a \to \mathcal{F}(\ell) \to 0.$$

Untwisting gives a surjection

$$\mathcal{O}_X(-\ell)^a \to \mathcal{F} \to 0.$$

Let $\mathcal{E} = \mathcal{O}_X(-\ell)^a$, then I claim that $\mathcal{E}$ is acyclic for both sides. First consider the left hand side. Then

$$\begin{aligned}\operatorname{Ext}^i(\oplus \mathcal{O}(-\ell), \mathcal{G}(n)) &= \oplus \operatorname{Ext}^i(\mathcal{O}_X(-\ell), \mathcal{G}(n)) \\ &= \oplus \operatorname{Ext}^i(\mathcal{O}_X, \mathcal{G}(\ell+n)) \\ &= H^i(X, \mathcal{G}(\ell+n))\end{aligned}$$

By Serre (theorem 5.2 of the book) this is zero for n sufficiently large. For the right hand side the statement is just that

$$\mathcal{E}\mathrm{xt}^i(\mathcal{E}, \mathcal{G}(n)) = 0$$

which we have already done since $\mathcal{E}$ is a locally free sheaf.

Thus both functors are universal since they are coeffaceable. Since universal functors are completely determined by their zeroth one they must be equal. □

Example 22.4. One might ask if Ext^i necessarily vanishes for sufficiently large i. The answer is no. Here is an algebraic example which can be converted to a geometric example. Let $A = k[\varepsilon]/(\varepsilon^2)$, then a projective resolution $L_\bullet$ of k is

$$\cdots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} k \to 0.$$

Then $Hom(L_\bullet, k)$ is the complex

$$k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} \cdots$$

Thus $\operatorname{Ext}^i_A(k,k) = k$ for all $i \geq 0$.

23 Serre Duality

We are now done with technical results on Ext's so we can get back to Serre duality on $\mathbf{P}^n$. Let $X = \mathbf{P}^n_k$ and let $\omega = \mathcal{O}_X(-n-1)$. Note that this is an *ad hoc* definition of ω which just happens to work since $X = \mathbf{P}^n_k$. In the more general situation it will be an interesting problem just to show the so called dualizing sheaf ω actually exists. When our variety is nonsingular, ω will be the canonical sheaf. We have shown that for any coherent sheaf $\mathcal{F}$ there is a map

$$\operatorname{Hom}(\mathcal{F}, \omega) \xrightarrow{\sim} H^n(\mathcal{F})^\vee.$$

The map is constructed by using the fact that H^n is a functor:

$$\operatorname{Hom}(\mathcal{F}, \omega) \to \operatorname{Hom}_k(H^n(\mathcal{F}), H^n(\omega)) = \operatorname{Hom}_k(H^n(\mathcal{F}), k) = H^n(\mathcal{F})^\vee.$$

We shall use satellite functors to prove the following theorem.

Theorem 23.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbf{P}^n_k$. Then there is an isomorphism*

$$\operatorname{Ext}^i(\mathcal{F}, \omega) \xrightarrow{\sim} H^{n-i}(\mathcal{F})^\vee.$$

Proof. Regard both sides as functors in $\mathcal{F}$.

1. Both sides are δ-functors in $\mathcal{F}$. We have already checked this for Ext^i. Since H^{n-i} is a delta functor in $\mathcal{F}$, so is $(H^{n-i})^\vee$. Note that both sides are contravarient.

2. They agree for $i = 0$. This was proved last time.

3. Now we just need to show both sides are coeffaceable. Suppose $\mathcal{E} \to \mathcal{F} \to 0$ with $\mathcal{E} = \mathcal{O}(-\ell)^{\oplus a}$. For some reason we can assume $\ell \gg 0$. We just need to show both sides vanish on this $\mathcal{E}$. First computing the left hand side gives

$$\oplus \operatorname{Ext}^i(\mathcal{O}(-\ell), \omega) = H^i(\omega(\ell)) = 0$$

for $\ell \gg 0$. Next computing the right hand side we get

$$H^{n-i}(\mathcal{O}(-\ell)) = 0$$

by the explicit computations of cohomology of projective space (in particular, note that $i > 0$). □

Next time we will generalize Serre duality to an arbitrary projective scheme X of dimension n. We will proceed in two steps. The first is to ask, what is ω_X? Although the answer to this question is easy on $\mathbf{P}^n_k$ it is not obvious what the suitable analogy should be for an arbitrary projective variety. Second we will define natural maps

$$\operatorname{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega) \xrightarrow{\varphi^i} H^{n-i}(\mathcal{F})^\vee$$

where $n = \dim X$. Unlike in the case when $X = \mathbf{P}^n_k$, these maps are not necessarily isomorphisms unless X is locally Cohen-Macaulay (the local rings at each point are Cohen-Macaulay).

Definition 23.2. Let A be a nonzero Noetherian local ring with residue field k. Then the **depth** of A is

$$\operatorname{depth} A = \inf\{i : \operatorname{Ext}^i_A(k, A) \neq 0\}.$$

A is said to be **Cohen-Macaulay** if $\operatorname{depth} A = \dim A$.

24 Serre Duality for Arbitrary Projective Schemes

Today we will talk about Serre duality for an arbitrary projective scheme. We have already talked about Serre duality in the special case $X = \mathbf{P}^n_k$. Let $\mathcal{F}$ be a locally free sheaf. We showed there is an isomorphism

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(\mathcal{F})^\vee.$$

This was established by noting that

$$\operatorname{Ext}^i(\mathcal{F}, \omega_X) = \operatorname{Ext}^i(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X) = H^i(\mathcal{F}^\vee \otimes \omega_X).$$

Another thing to keep in mind is that locally free sheaves correspond to what, in other branches of mathematics, are vector bundles. They aren't the same object, but there is a correspondence.

We would like to generalize this to an arbitrary projective scheme X. There are two things we must do.

1. Figure out what ω_X is.

2. Prove a suitable duality theorem.

When $X = \mathbf{P}^n_k$ it is easy to find a suitable $\omega_X = \mathcal{O}_{\mathbf{P}^n_k}(-n-1)$ because of the explicit computations we did before. We now define ω_X to be a sheaf which will do what we hope it will do. Of course existence is another matter.

Definition 24.1. Let X be a Noetherian scheme of finite type over a field k and let $n = \dim X$. Then a **dualizing sheaf** for X is a coherent sheaf ω_X along with a map $t : H^n(\omega_X) \to k$, such that for all coherent sheaves $\mathcal{F}$ on X, the map $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \to H^n(\mathcal{F})^\vee$ is an isomorphism. The latter map is defined by the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) & & \\ \downarrow & & \\ \mathrm{Hom}(H^n(\mathcal{F}), H^n(\omega_X)) & \xrightarrow{t} & \mathrm{Hom}(H^n(\mathcal{F}), k) = H^n(\mathcal{F})^\vee \end{array}$$

Strictly speaking, a dualizing sheaf is a pair (ω_X, t). Note that on $\mathbf{P}^n$ we had $H^n(\omega_{\mathbf{P}^n}) \cong k$, but on an arbitrary scheme X we only have a map from $H^n(\omega_X)$ to k which need not be an isomorphism. The definition never mentions existence.

Proposition 24.2. *If X admits a dualizing sheaf (ω_X, t) then the pair (ω_X, t) is unique up to unique isomorphism, i.e., if (η, s) is another dualizing sheaf for X then there is a unique isomorphism $\varphi : \omega_X \to \eta$ such that*

$$\begin{array}{ccc} H^n(\omega_X) & \xrightarrow{H^n(\varphi)} & H^n(\eta) \\ t \downarrow & & \downarrow s \\ k & = & k \end{array}$$

commutes.

Before we prove the proposition we make a short digression to introduce representable functors which give a proof of the uniqueness part of the above proposition.

Definition 24.3. Let $\mathcal{C}$ be a category and $\mathcal{D}$ a category whose objects happen to be sets. Suppose $T : \mathcal{C} \to \mathcal{D}$ is a contravarient functor. Then T is **representable** if there exists an object $\omega \in \mathrm{Ob}(\mathcal{C})$ and an element $t \in T(\omega)$ such that for all $F \in \mathrm{Ob}(\mathcal{C})$ the map $\mathrm{Hom}_{\mathcal{C}}(F, \omega) \to T(F)$ is a bijection of sets. The latter map is defined by the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}}(F, \omega) & & \xrightarrow{\text{bijection of sets}} & & T(F) \\ & \searrow & & \nearrow \text{ evaluation at } t & \\ & & \mathrm{Hom}_{\mathcal{D}}(\Gamma(\omega), T(F)) & & \end{array}$$

Thus there is an isomorphism of functors $\mathrm{Hom}(\bullet, \omega) = T(\bullet)$. The pair (t, ω) is said to represent the functor T. The relevant application of this definition is to the case when $\mathcal{C} = \mathbf{Coh}(X)$, $\mathcal{D} = \{\ k\text{-vector spaces}\}$, T is the functor $F \mapsto H^n(\mathcal{F})^\vee$. Then $\omega = \omega_X$ and

$$t = t \in \mathrm{Hom}(H^n(\omega), k) = H^n(\omega)^\vee = T(\omega).$$

Proposition 24.4. *If T is a representable functor, then the pair (ω, t) representing it is unique.*

Proof. Suppose (ω, t) and (η, s) both represent the functor T. Consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\eta, \omega) & \xrightarrow{T} & \mathrm{Hom}(T(\omega), T(\eta)) \\ & \searrow & \swarrow \text{ eval. at } t \\ & T(\eta) & \end{array}$$

By definition the map $\mathrm{Hom}(\eta, \omega) \to T(\eta)$ is bijective. Since $s \in T(\eta)$, there is $\varphi \in \mathrm{Hom}(\eta, \omega)$ such that $\varphi \mapsto s \in T(\eta)$. Thus φ has the property that $T(\varphi)(t) = s$. This argument uses the fact that (ω, t) represents T. Using the fact that (η, s) represents T implies that there exists $\psi \in \mathrm{Hom}(\omega, \eta)$ such that $T(\psi)(s) = t$. We have the following pictures

$$\omega \underset{\phi}{\overset{\psi}{\rightleftarrows}} \eta$$

$$t \in T(\omega) \underset{T(\phi)=\phi^*}{\overset{T(\psi)=\psi^*}{\leftrightarrows}} T(\eta) \ni s$$

I claim that

$$\psi \circ \varphi = \mathrm{Id} \in \mathrm{Hom}(\eta, \eta).$$

In diagram form we have

$$\eta \xrightarrow{\varphi} \omega \xrightarrow{\psi} \eta$$

which upon applying T gives

$$\begin{gathered} T(\eta) \xrightarrow{\psi^*} T(\omega) \xrightarrow{\varphi^*} T(\eta) \\ s \mapsto t \mapsto s \end{gathered}$$

Where does $\psi \circ \varphi$ go to under the map $\mathrm{Hom}(\eta, \eta) \xrightarrow{\sim} T(\eta)$? By definition $\psi \circ \varphi$ goes to the evaluation of $T(\psi \circ \varphi)$ at $s \in T(\eta)$. But, as indicated above, the evaluation of $T(\psi \circ \varphi)$ at s is just s again. But the identity morphism $1_\eta \in \mathrm{Hom}(\eta, \eta)$ also maps to s under the map $\mathrm{Hom}(\eta, \eta) \xrightarrow{\sim} T(\eta)$. Since this map is a bijection this implies that $\psi \circ \varphi = 1_\eta$, as desired. Similarly $\varphi \circ \psi = 1_\omega$. Thus ψ and φ are both isomorphisms. $\square$

> "When you define something and it is unique up to unique isomorphism, you know it must be good."

We return to the question of existence.

Proposition 24.5. *If X is a projective scheme over a field k then (ω_X, t) exists.*

Lemma 24.6. *If X is an n dimensional projective scheme over a field k, then there is a finite morphism $f : X \to \mathbf{P}^n_k$.*

Proof. Embed X in $\mathbf{P}^N$ then choose a linear projection down to $\mathbf{P}^n$ which is sufficiently general.

$$\begin{array}{ccc} X & \hookrightarrow & \mathbf{P}^N \\ & f \searrow & \downarrow \\ & & \mathbf{P}^n \end{array}$$

Let L be a linear space of dimension $N - n - 1$ not meeting X. Let the map from $\mathbf{P}^N \to \mathbf{P}^n$ be projection through L. By construction f is quasi-finite, i.e., for all $Q \in \mathbf{P}^n$, $f^{-1}(Q)$ is finite. It is a standard QUALIFYING EXAM problem to show that if a morphism is quasi-finite and projective then it is finite. This can be done by applying (II, Ex. 4.6) by covering X by subtracting off hyperplanes and noting that the correct things are affine by construction. See also (III, Ex. 11.2) for the more general case when f is quasi-finite and proper, but not necessarily projective. $\square$

25 Existence of the Dualizing Sheaf on a Projective Scheme

Let X be a scheme over k. Recall that a dualizing sheaf is a pair (ω, t) where ω is a coherent sheaf on X and

$$t : H^n(X, \omega) \to k$$

is a homomorphism such that for all coherent sheaves $\mathcal{F}$ the natural map

$$\mathrm{Hom}_X(\mathcal{F}, \omega) \to H^n(\mathcal{F})^\vee$$

is an isomorphism. We know that such a dualizing sheaf exists on $\mathbf{P}^n_k$.

Theorem 25.1. *If X is a projective scheme of dimension n over k, then X has a dualizing sheaf.*

The book's proof takes an embedding $j : X \hookrightarrow \mathbf{P}^N_k$ and works on X as a subscheme of $\mathbf{P}^N_k$. Then the book's proof shows that

$$\omega_X = \mathcal{E}\mathrm{xt}^{N-n}_{\mathcal{O}_{\mathbf{P}^N_k}}(\mathcal{O}_X, \omega_{\mathbf{P}^N_k}).$$

Today we will use a different method.

Definition 25.2. A **finite morphism** is a morphism $f : X \to Y$ of Noetherian schemes such that for any open affine $U = \mathrm{Spec}\, A \subset X$, the preimage $f^{-1}(U) \subset Y$ is affine, say $f^{-1}(U) = \mathrm{Spec}\, B$, and the natural map $A \to B$ turns B into a finitely generated A-module. We call f an **affine morphism** if we just require that $f^{-1}(U)$ is affine but not that B is a finitely generated A-module. A morphism $f : X \to Y$ is **quasi-finite** if for all $y \in Y$ the set $f^{-1}(y)$ is finite.

Example 25.3. Consider the morphism

$$j : \mathbf{P}^1 - \{\mathrm{pt}\} \hookrightarrow \mathbf{P}^1.$$

Since $\mathbf{P}^1$ minus any nonempty finite set of points is affine j is affine. But it is not finite. Indeed, let a be a point different from pt and let $U = \mathbf{P}^1 - \{a\}$. Then $U = \mathrm{Spec}\, k[x]$ and

$$j^{-1}(U) = \mathbf{P}^1 - \{\mathrm{pt}, a\} = \mathrm{Spec}\, k[x, x^{-1}],$$

but $k[x, x^{-1}]$ is not a finitely generated $k[x]$-module.

Exercise 25.4. A morphism can be affine but not finite or even quasi-finite. For example, let f be the natural map

$$f : \mathbf{A}^{n+1} - \{0\} \to \mathbf{P}^n$$

then show that f is affine. This is the fiber bundle associated to the invertible sheaf $\mathcal{O}(1)$ [[or is it $\mathcal{O}(-1)$?]]

25.1 Relative Gamma and Twiddle

We will now define relative versions of global sections and ˜ analogous to the absolute versions. It is not a generalization of the absolute notion, but a relativization. Suppose X is a scheme over Y with structure map $f : X \to Y$ and assume f is affine. Then the map sending a sheaf $\mathcal{F}$ on X to the sheaf $f_*\mathcal{F}$ on Y is the analog of taking global sections. Since f is a morphism there is a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ so $f_*\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-modules. Note that $f_*\mathcal{F}$ is a sheaf of $f_*\mathcal{O}_X$-modules. Thus we have set up a map

$$\mathbf{Qco}(X) \to \{ \text{ quasicoherent } f_*\mathcal{O}_X\text{-modules on } Y \}.$$

The next natural thing to do is define a map analogous to $\tilde{}$ which goes the other direction. Suppose $\mathcal{G}$ is a quasi coherent sheaf of $f_*\mathcal{O}_X$-modules on Y. Let $U \subset Y$ be an affine open subset of Y. Let $G = \Gamma(U, \mathcal{G})$ and write $U = \operatorname{Spec} A$. Then since f is an affine morphism, $f^{-1}(U) = \operatorname{Spec} B$ where $B = \Gamma(f^{-1}(U), \mathcal{O}_X)$. Since $\mathcal{G}$ is an $f_*\mathcal{O}_X$-module, and $f_*\mathcal{O}_X$ over U is just B thought of as an A-module, we see that G is a B-module. Thus we can form the sheaf $\tilde{G}$ on $\operatorname{Spec} B = f^{-1}(U)$. Patching the various sheaves $\tilde{G}$ together as U runs through an affine open cover of Y gives a sheaf $\tilde{\mathcal{G}}$ in $\mathbf{Qco}(X)$.

Let $\mathcal{G}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-modules. We can't take $\tilde{}$ of $\mathcal{G}$ because $\mathcal{G}$ might not be a sheaf of $f_*\mathcal{O}_X$-modules. But $\mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})$ is a sheaf of $f_*\mathcal{O}_X$-modules, so we can form $(\mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G}))\tilde{}$. This is a quasi-coherent sheaf on X which we denote $f^!(\mathcal{G})$.

Proposition 25.5. *Suppose $f : X \to Y$ is an affine morphism of Noetherian schemes, $\mathcal{F}$ is coherent on X, and $\mathcal{G}$ is quasi-coherent on Y. Then*

$$f_* \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G})$$

and passing to global sections gives an isomorphism

$$\operatorname{Hom}(\mathcal{F}, f^!\mathcal{G}) \cong \operatorname{Hom}(f_*\mathcal{F}, \mathcal{G}).$$

Thus $f^!$ is a right adjoint for f_*.

Proof. The *natural* map is

$$\begin{aligned} f_* \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) &\to \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{F}, f_*f^!\mathcal{G}) \\ &= \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G}) \end{aligned}$$

where the map $\mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G}) \to \mathcal{G}$ is obtained obtained by evaluation at 1. Since the question is local we may assume $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$. Then $\mathcal{F}$ corresponds to a finitely generated module M over the Noetherian ring B and $\mathcal{G}$ corresponds to a module N over A. We must show that

$$\operatorname{Hom}_B(M, \operatorname{Hom}_A(B, N)) \cong \operatorname{Hom}_A(M, N).$$

When M is free over B so that $M = B^{\oplus n}$ the equality holds. As functors in M, both sides are contravarient and left exact. Now suppose M is an arbitrary finitely generated B-module. Write M as a quotient F_0/F_1 where F_0 and F_1 are both free of finite rank. Applying each of the contravarient left-exact functors to the exact sequence

$$F_1 \to F_0 \to M \to 0$$

and using the fact that equality holds for finite free modules yields a diagram

$$\begin{array}{ccccccc} 0 & \to & \operatorname{Hom}_B(M, \operatorname{Hom}_A(B, N)) & \to & \operatorname{Hom}_B(F_0, \operatorname{Hom}_A(B, N)) & \to & \operatorname{Hom}_B(F_1, \operatorname{Hom}_A(B, N)) \\ & & & & \| & & \| \\ 0 & \to & \operatorname{Hom}_A(M, N) & \to & \operatorname{Hom}_A(F_0, N) & \to & \operatorname{Hom}_A(F_1, N) \end{array}$$

The 5-lemma then yields an isomorphism

$$\operatorname{Hom}_B(M, \operatorname{Hom}_A(B, N)) \cong \operatorname{Hom}_A(M, N).$$

$\square$

Lemma 25.6. *Suppose $f : X \to Y$ is affine and $\mathcal{F}$ is a quasi-coherent sheaf on X. Then*

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

Proof. The lemma is proved using Čech cohomology. If $\{U_i\}$ is an open affine cover of Y then $\{f^{-1}(U_i)\}$ is an open affine cover of X. But

$$\Gamma(U_i, f_*\mathcal{F}) = \Gamma(f^{-1}(U_i), \mathcal{F})$$

so the Čech cohomology of $f_*\mathcal{F}$ on Y is the same as the Čech cohomology of $\mathcal{F}$ on X. $\square$

Theorem 25.7 (Duality for a finite flat morphism). *Suppose $f : X \to Y$ is a finite morphism with X and Y Noetherian and assume every coherent sheaf on X is the quotient of a locally free sheaf (this is true for almost every scheme arising naturally in this course). Assume that $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module. Let $\mathcal{F}$ be a coherent sheaf on X and $\mathcal{G}$ be a quasi-coherent sheaf on Y. Then for all $i \geq 0$ there is an isomorphism*

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{G}).$$

Proof. The proposition shows that the theorem is true when $i = 0$. We next show that the statement is true when $\mathcal{F} = \mathcal{O}_X$. Applying the above lemma we see that

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{O}_X, f^!\mathcal{G}) &= H^i(X, f^!\mathcal{G}) = H^i(Y, f_*f^!\mathcal{G}) \\ &= H^i(Y, \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G}))\end{aligned}$$

Since $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module,

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G}) &= \mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{O}_Y, (f_*\mathcal{O}_X)^\vee \otimes \mathcal{G}) \\ &= H^i(Y, (f_*\mathcal{O}_X)^\vee \otimes \mathcal{G}))\end{aligned}$$

Putting these two computations together by using the fact that

$$(f_*\mathcal{O}_X)^\vee \otimes \mathcal{G} = \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})$$

then gives the desired result.

A clever application of the 5-lemma can be used to obtain the general case. This will be done in a subsequent lecture. $\square$

26 Generalized Grothendieck Duality Theory

If X is a projective scheme over k, then

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(\mathcal{F})^\vee.$$

This is a special type of duality. If X is an affine scheme over Y with structure morphism $f : X \to Y$, then

$$\mathrm{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\sim} \mathrm{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G}).$$

This is another duality.

At Harvard, Hartshorne was the scribe [Lecture Notes in Math Vol ???] for Grothendieck's seminar on his duality theory. Suppose X is proper over Y with structure morphism $f : X \to Y$. Assume furthermore that f satisfies hypothesis (*):

$$(*) \quad f_*\mathcal{O}_X \text{ is a locally free } \mathcal{O}_Y\text{-module}$$

Note that hypothesis start implies $\mathcal{H}\mathrm{om}(f_*\mathcal{O}_X, \mathcal{G})$ is an exact functor in $\mathcal{G}$. We have defined functors $f_* : \mathbf{Coh}(X) \to \mathbf{Coh}(Y)$ and $f^! : \mathbf{Qco}(Y) \to \mathbf{Qco}(X)$. Grothendieck's duality compares $\mathrm{Hom}_X(\mathcal{F}, f^!\mathcal{G})$ and its derived functors to $\mathrm{Hom}_Y(f_*\mathcal{F}, \mathcal{G})$ and its derived functors. The two families of functors are essentially equal. We are composing functors here. This often gives rise to spectral sequences.

It is possible to obtain the duality mentioned above as a special case of the more general Grothendieck duality. Suppose X is a projective scheme over $Y = \operatorname{Spec} k$ with morphism $f : X \to Y$. Then $f_*(\mathcal{F}) = \Gamma(X, \mathcal{F})$ which has derived functors $H^i(X, \bullet)$. A coherent sheaf on Y is a finite dimensional k-vector space. Let ω_X be the sheaf $f^!(k)$. Substituting this into

$$\mathrm{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \xrightarrow{\sim} \mathrm{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G})$$

with $G = k$ gives

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \xrightarrow{\sim} \mathrm{Ext}^i_Y(f_*\mathcal{F}, k) = H^{n-i}(\mathcal{F})^\vee$$

[[Do we obtain H^{n-i} instead of H^i since we are considering the derived functors of $\mathrm{Hom}(f^*(\bullet), \bullet)$?]]

27

Theorem 27.1 (Duality for a projective scheme). *Suppose X is a projective scheme. Then X has a dualizing sheaf ω_X and their is an isomorphism*

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

Proof. Using linear projections find a finite morphism $f : X \to \mathbf{P}^n$. Let $\omega_X = f^!\omega_{\mathbf{P}^n}$ where

$$f^!\omega_{\mathbf{P}^n} = \mathcal{H}\mathrm{om}_{\mathbf{P}^n}(f_*\mathcal{O}_X, \omega_{\mathbf{P}^n})\tilde{}$$

and $\omega_{\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^n}(-n-1)$. Then for any $\mathcal{F}$,

$$\begin{aligned} \mathrm{Hom}_X(\mathcal{F}, \omega_X) &= \mathrm{Hom}_{\mathbf{P}^n}(f_*\mathcal{F}, \omega_{\mathbf{P}^n}) \\ &\xrightarrow{\sim} H^n(\mathbf{P}^n, f_*\mathcal{F})^\vee = H^n(X, \mathcal{F})^\vee \end{aligned}$$

The last equality holds since f is finite and hence affine. The second isomorphism comes from the fact that ω_X is a dualizing sheaf for X.

Next we obtain the duality theorem for X. By duality for a finite (*) morphism

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \cong \mathrm{Ext}^i_{\mathbf{P}^n}(f_*\mathcal{F}, \omega_{\mathbf{P}^n}).$$

By Serre duality on $\mathbf{P}^n$

$$\mathrm{Ext}^i_{\mathbf{P}^n}(f_*\mathcal{F}, \omega_{\mathbf{P}^n}) \xrightarrow{\sim} H^{n-i}(\mathbf{P}^n, f_*\mathcal{F})^\vee.$$

But f is affine so

$$H^{n-i}(\mathbf{P}^n, f_*\mathcal{F})^\vee = H^{n-i}(X, \mathcal{F})^\vee.$$

Thus

$$\mathrm{Ext}^i_X(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^\vee.$$

□

Under what conditions does a finite map $f : X \to \mathbf{P}^n$ satisfy (*)?

Definition 27.2. A scheme X is **Cohen-Macaulay** if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay.

Definition 27.3. The **homological dimension** of a module M over a ring A is the minimum possible length of a projective resolution of M in the category of A-modules. We denote this number by $\operatorname{hd} M$.

We will need the following theorem from pure algebra.

Theorem 27.4. *If A is a regular local ring and M a finitely generated A-module, then*

$$\operatorname{hd} M + \operatorname{depth} M = \dim A.$$

Theorem 27.5. *Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes and assume Y is nonsingular. Then f satisfies (*) iff X is Cohen-Macaulay.*

Proof. The question is local on Y. Suppose $y \in Y$, then $A = \mathcal{O}_{Y,y}$ is a regular local ring and $f^{-1}(y) \subset X$ is a finite set. Let B be the semi local ring of $f^{-1}(y)$. Thus $B = \varinjlim_U \mathcal{F}(U)$ where the injective limit is taken over all open sets U containing $f^{-1}(y)$. B has only finitely many maximal ideals so we call B semi local. Since f is a finite morphism B is a finite A-module. Now B is free as an A-module iff $\operatorname{hd}_A B = 0$, where $\operatorname{hd}_A B$ denotes the homological dimension of B as an A-module. This is clear because $\operatorname{hd}_A B$ is the shortest possible length of a free resolution of B. (Since A is local we need only consider free resolutions and not the more general projective resolutions.) Now $(f_*\mathcal{O}_X)_y = B$ so condition (*) is that B is a free A-module. Thus if f satisfies (*) then $\operatorname{hd} B = 0$. By the theorem from pure algebra and the fact that f is finite we see that

$$\operatorname{depth} B = \dim A = \dim B.$$

This implies B is a Cohen-Macaulay ring and therefore X is Cohen-Macaulay. Conversely, if X is Cohen-Macaulay then B is Cohen-Macaulay so $\operatorname{depth} B = \dim B$. The purely algebraic theorem then implies that $\operatorname{hd} B = 0$ so B is a free A-module and hence f satisfies condition (*). [[We are tacitly assuming that B is Cohen-Macaulay iff it's localizations at maximal ideals are. It would be nice to know this is true.]] $\square$

Now we finish up the proof of duality for a finite morphism.

Proof (of duality, continued). Suppose $f : X \to Y$ is a finite morphism of Noetherian schemes which satisfies (*), and assume that X has enough locally free sheaves (i.e. every coherent sheaf is a quotient of a locally free sheaf). We showed that

$$\operatorname{Hom}_X(\mathcal{F}, f^!\mathcal{G}) \cong \operatorname{Hom}_Y(f_*\mathcal{F}, \mathcal{G}).$$

This is the $i = 0$ case of the theorem and is true even if we drop the assumption that f satisfies (*). It can be shown by taking an injective resolution of $f^!\mathcal{G}$ and computing Ext^i_X using it that there are natural maps

$$\varphi^i : \operatorname{Ext}^i_X(\mathcal{F}, f^!\mathcal{G}) \to \operatorname{Ext}^i_Y(f_*\mathcal{F}, \mathcal{G}).$$

We have already shown that this is an isomorphism when $sF = \mathcal{O}_X$. By the same argument one shows that this is an isomorphism when $\mathcal{F}$ is just locally free. The key point to notice is that if $\mathcal{F}$ is locally free then $\mathcal{F}$ is locally free over $f_*\mathcal{O}_X$ which is locally free over $\mathcal{O}_Y$

by condition (*). Finally suppose $\mathcal{F}$ is an arbitrary coherent sheaf on X. Writing $\mathcal{F}$ as a quotient of of a locally free sheaf $\mathcal{E}$ gives an exact sequence

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{R}$ coherent. The long exact sequence of Ext's gives a diagram

$$\begin{array}{ccccccccc} \mathrm{Hom}_X(\mathcal{E}, f^!\mathcal{G}) & \to & \mathrm{Hom}_X(\mathcal{R}, f^!\mathcal{G}) & \to & \mathrm{Ext}^1_X(\mathcal{F}, f^!\mathcal{G}) & \to & \mathrm{Ext}^1(\mathcal{E}, f^!\mathcal{G}) & \to & \mathrm{Ext}^1_X(\mathcal{R}, f^!\mathcal{G}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow ?? & & \downarrow\cong & & \downarrow ?? \\ \mathrm{Hom}_X(f_*\mathcal{E}, \mathcal{G}) & \to & \mathrm{Hom}_X(f_*\mathcal{R}, \mathcal{G}) & \to & \mathrm{Ext}^1_X(f_*\mathcal{F}, \mathcal{G}) & \to & \mathrm{Ext}^1(f_*\mathcal{E}, \mathcal{G}) & \to & \mathrm{Ext}^1_X(f_*\mathcal{R}, \mathcal{G}) \end{array}$$

Now apply the *subtle* 5-lemma to show that the map

$$\varphi^1 : \mathrm{Ext}^1_X(\mathcal{F}, f^!\mathcal{G}) \to \mathrm{Ext}^1_X(f_*\mathcal{F}, \mathcal{G})$$

is injective. Another diagram chase shows that he map

$$\mathrm{Ext}^1_X(\mathcal{R}, f^!\mathcal{G}) \to \mathrm{Ext}^1_X(f_*\mathcal{R}, \mathcal{G})$$

is injective. Then [[I guess??]] the 5-lemma shows that φ^i is an isomorphism. Climbing the sequence inductively shows that φ^i is an isomorphism for all i. $\square$

Exercise 27.6. Give an example of a scheme X which does not have enough locally free sheaves. [Hints: By (III, 6.8) X has enough locally free sheaves if X is quasi-projective or nonsingular. Hartshorne intimated that this problem is *hard*, but suggested one might search for a counterexample by looking for an appropriate non-projective 3-fold with 2 singular points.]

28 Review of Differentials

Definition 28.1. Let B be a ring and M a B-module. Then a map $d : B \to M$ is a **derivation** if

$$d(b_1 + b_2) = db_1 + db_2$$
$$d(b_1 b_2) = b_1 db_2 + b_2 db_1$$

If B is an A-algebra, then d is a **derivation over** A if in addition $da = 0$ for all $a \in A$.

Note that a derivation over A is linear since $d(ab) = adb + bda = adb + 0 = adb$.

Example 28.2. Let k be a field and let $B = k[x]$. Let $d : B \to B$ be the differentian map $f(x) \mapsto f'(x)$. Then d is a derivation.

Definition 28.3. Let B be an A-algebra. Then the **module of differentials** of B over A is a pair $(\Omega_{B/A}, d : B \to \Omega_{B/A})$, where $\Omega_{B/A}$ is a B-module and d is a derivation of B into $\Omega_{B/A}$ over A, which satisfies the following universal property: if $d' : B \to M$ is a derivation over A then there exists a unique B-linear map $\varphi : \Omega_{B/A} \to M$ such that $d' = \varphi \circ d$.

If $(\Omega_{B/A}, d)$ exists it is unique as a pair up to unique isomorphism.

We first construct $\Omega_{B/A}$ by brute force. Let $\Omega_{B/A}$ be the free module on symbols db (all $b \in B$) modulo the submodule generated by the relations $d(b_1 + b_2) - db_1 - db_2$, $d(b_1 b_2) - b_2 db_1 - b_1 db_2$, and da for all $b_1, b_2 \in B$ in $a \in A$. This is obviously a module of differentials.

Example 28.4. Suppose $A = B$, then $\Omega_{B/A} = 0$.

Example 28.5. Suppose k is a field of characteristic $p > 0$. Let $B = k[x]$ and let $A = k[x^p]$. Then $\Omega_{B/A}$ is the free B-module of rank 1 generated by dx.

Example 28.6. Let $B = \mathbf{Q}[\sqrt{2}]$ and $A = \mathbf{Q}$. Then $0 = d(2) = d(\sqrt{2}\sqrt{2}) = 2\sqrt{2}d(\sqrt{2})$ so $d(\sqrt{2}) = 0$. Thus $\Omega_{\mathbf{Q}[\sqrt{2}]/\mathbf{Q}} = 0$.

Corollary 28.7. *The module of differentials $\Omega_{B/A}$ is generated by $\{db : b \in B, b \notin A\}$.*

Now we will construct $\Omega_{B/A}$ in a more eloquent manner. Suppose B is an A-algebra. Consider the exact sequence of A-modules

$$0 \to I \to B \otimes_A B \xrightarrow{\Delta} B \to 0$$

where Δ is the diagonal map $b_1 \otimes b_2 \mapsto b_1 b_2$ and I is the kernel of Δ. Make I/I^2 into a B-module by letting B act on the first factor (thus $b(x \otimes y) = (bx) \otimes y$) and define a map $d : B \to I/I^2$ by $db = 1 \otimes b - b \otimes 1$.

Proposition 28.8. *The module I/I^2 along with the map d is the module of differentials for B over A.*

Proof. Suppose $b_1, b_2 \in B$, then

$$d(b_1 b_2) = 1 \otimes b_1 b_2 - b_1 b_2 \otimes 1.$$

One the other hand,

$$\begin{aligned} b_1 db_2 + b_2 db_1 &= b_1(1 \otimes b_2 - b_2 \otimes 1) + b_2(1 \otimes b_1 - b_1 \otimes 1) \\ &= b_1 \otimes b_2 - b_1 b_2 \otimes 1 + b_2 \otimes b_1 - b_1 b_2 \otimes 1 \end{aligned}$$

Now taking the difference gives

$$\begin{aligned} 1 \otimes b_1 b_2 - b_1 b_2 \otimes 1 - b_1 \otimes b_2 + b_1 b_2 \otimes 1 - b_2 \otimes b_1 + b_1 b_2 \otimes 1 \\ = (1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1) \in I^2 \end{aligned}$$

For the universal property see Matsumura. □

Corollary 28.9. *If B is a finitely generated A-algebra and A is Noetherian then $\Omega_{B/A}$ is a finitely generated B-module.*

Proof. If B is a finitely generated A-algebra then B is a Noetherian ring. Since I is a kernel of a ring homomorphism I is an ideal so I is finitely generated. Thus I/I^2 is finitely generated as a B-module. □

Example 28.10. Let A be any ring and let $B = A[x_1, \ldots, x_n]$. Then $\Omega_{B/A}$ is the free B-module generated by $dx_1, \ldots, dx_n$. The derivation $d : B \to \Omega_{B/A}$ is the map $f \mapsto \sum \frac{\partial f}{\partial x_i} dx_i$. Since B is generated as an A-algebra by the x_i, $\Omega_{B/A}$ is generated as a B-module by the dx_i and there is an epimorphism $B^r \to \Omega_{B/A}$ taking the ith basis vector to dx_i.

On the other hand, the partial derivative $\partial/\partial x_i$ is an A-linear derivation from B to B, and thus induces a B-module map $\partial_i : \Omega_{B/A} \to B$ carrying dx_i to 1 and all the other x_j to 0. Putting these maps together we get the inverse map. This proof is lifted from Eisenbud's *Commutative Algebra*.

There are a few nice exact sequences.

Proposition 28.11. *Suppose A, B, and C are three rings and*

$$A \to B \xrightarrow{g} C$$

is a sequence of maps between them (it needn't be exact – in fact it wouldn't make sense to stipulate that it is exact because kernels don't exist in the category of commutative rings with 1). Then there is an exact sequence of C-modules

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

If $d : B \to \Omega_{B/A}$ is the derivation associated with $\Omega_{B/A}$ then the first map is $db \otimes_B c \mapsto cd(g(b))$.

We won't prove this here, but note that exactness in the middle is the most interesting. The functors T^i comes next on the left. See the work of Schlesinger and Lichenbaum, or Illusi and André.

Proposition 28.12. *Suppose A, B, and C are rings and*

$$A \to B \to C$$

is a sequence of maps. Assume furthermore $I \subset B$ is an ideal, that $C = B/I$, and the map from $B \to C$ is the natural surjection. Then there is an exact sequence of C-modules

$$I/I^2 \xrightarrow{d} \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} = 0$$

Next we consider what happens for a local ring.

Proposition 28.13. *Suppose $B, \mathbf{m}$ is a local ring with residue field $k = B/\mathbf{m}$ and there is an injection $k \hookrightarrow B$. Then there is an exact sequence*

$$\mathbf{m}/\mathbf{m}^2 \xrightarrow{d} \Omega_{B/k} \otimes_B k \to 0$$

and d is actually an isomorphism.

The exact sequence is obtained from the previous proposition by letting $A = C = k$. The fact that d is an isomorphism is supposed to be tricky. The proposition implies that a lower bound on the number of generators of $\Omega_{B/k}$ is $\dim_k \mathbf{m}/\mathbf{m}^2$. If B is regular and local, then $\dim B = \dim_k \mathbf{m}/\mathbf{m}^2$ which implies $\Omega_{B/k}$ can be generated by $\dim_k B$ elements.

Proposition 28.14. *Let B be a localization of an algebra of finite type over a perfect field, let $\mathbf{m}$ be the maximal ideal and let $k = B/\mathbf{m}$. Then B is a regular local ring iff $\Omega_{B/k}$ is a free B-module of rank equal to the dimension of B over k.*

Proof. ($\leftarrow$) If $\Omega_{B/k}$ is free of rank $n = \dim_k B$ then the minimum number of generators of $\Omega_{B/k}$ is n, so $\dim \Omega_{B/k} \otimes k = n = \dim_k \mathbf{m}/\mathbf{m}^2$. Thus $\dim B = \dim \mathbf{m}/\mathbf{m}^2$ whence B is regular. ($\rightarrow$) ∎

28.1 The Sheaf of Differentials on a Scheme

Suppose $f : X \to Y$ is a morphism of schemes. Let $V = \operatorname{Spec} B$ be an open affine subset of X and $U = \operatorname{Spec} A$ an open affine subset of Y such that $f(V) \subset U$. Then B is an algebra over A so we may consider the module $\Omega_{B/A}$. Put $\widetilde{\Omega_{B/A}}$ on V and glue to get a sheaf $\Omega_{X/Y}$. We can glue because localization commutes with forming Ω and the universal property of Ω makes gluing isomorphisms canonical and unique.

If X/k is a nonsingular variety of dimension n then $\Omega_{X/k}$ is locally free of rank n. If X is a curve, then $\Omega_{X/k}$ is locally free of rank 1 so it is a line bundle.

The sheaf $\Omega_{X/k}$ is important because it is intrinsically defined and canonically associated to $X \xrightarrow{f} Y$.

29 Differentials on $\mathbf{P}^n$

Remark 29.1. Suppose X is a scheme over Y then one could also define $\Omega_{X/Y}$ as follows. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism and let $\mathcal{I}_\Delta$ be the ideal sheaf of the image of Δ. Define $\Omega_{X/Y} = \Delta^*(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2)$.

Proposition 29.2. *Let X, Y, and Z be schemes along with maps $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then there is an exact sequence of $\mathcal{O}_X$-modules*

$$f^*(\Omega_{Y/Z}) \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Proposition 29.3. *Let Y be a closed subscheme of a scheme X over S. Let $\mathcal{I}_Y$ be the ideal sheaf of Y on X. Then there is an exact sequence of sheaves of $\mathcal{O}_X$-modules*

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_{X/S} \otimes \mathcal{O}_Y \to \Omega_{Y/S} \to 0.$$

Example 29.4. Let S be a scheme and let $X = \mathbf{A}^n_S$ be affine n-space over S. Then $\Omega_{X/S}$ is the free $\mathcal{O}_X$-module generated by $dx_1, \dots, dx_n$.

Projective space is more interesting.

Theorem 29.5. *Let $X = \mathbf{P}^n_k$, then there is an exact sequence*

$$0 \to \Omega_{X/k} \to \mathcal{O}_X(-1)^{n+1} \to \mathcal{O}_X \to 0.$$

The proof in the book is too computational and nobody understands it therefore we present an "explanation" even though it doesn't quite have the force of proof.

Explanation. Since $U_i = \{x_i \neq 0\}$ is affine, $\Omega_X|_{U_i} = \Omega_{U_i}$ is free of rank n thus Ω_X is locally free of rank n. Let $W = \mathbf{A}^{n+1} - \{0\}$ and let $f : W \to X$ be the natural quotient map. The sequence $W \xrightarrow{f} X \to k$ gives rise to an exact sequence

$$f^*\Omega_X \to \Omega_W \to \Omega_{W/X} \to 0.$$

Since the open subset $W \subset \mathbf{A}^{n+1}$ is a nonsingular variety, Ω_W is free of rank $n+1$, generated by $dx_0, \dots, dx_n$.

The affine subset $U_0 = \{x_0 \neq 0\} \subset X$ can be represented as

$$U_0 = \operatorname{Spec} k[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}] = \operatorname{Spec} k[y_1, \dots, y_n].$$

The inverse image is

$$\begin{aligned} f^{-1}(U_0) = \mathbf{A}^{n+1} - \{x_0 = 0\} &= \operatorname{Spec}(k[x_0, \dots, x_n, \frac{1}{x_0}]) \\ &= \operatorname{Spec}(k[y_1, \dots, y_n][x_0, \frac{1}{x_0}]) \end{aligned}$$

Thus $f^{-1}(U_0) \cong U_0 \times (\mathbf{A}^1 - \{0\})$ which is affine. Therefore $\Omega_{W/X}|_{f^{-1}(U_0)}$ is locally free of rank 1 generated by dx_0. Consider again the exact sequence

$$f^*\Omega_X \to \Omega_W \to \Omega_{W/X} \to 0.$$

Since Ω_W is free of rank $n+1$, $\Omega_{W/X}$ is locally free of rank 1, and $f^*\Omega_X$ is locally free of rank n (pullback preserves rank locally), we conclude that the map $f^*\Omega_X \to \Omega_W$ must be injective. We thus obtain an exact sequence of sheaves on W

$$0 \to f^*\Omega_X \to \Omega_W \to \Omega_{W/X} \to 0.$$

Passing to global sections and using the fact that W is affine we obtain an exact sequence of modules over $S = k[x_0, \dots, x_n]$

$$0 \to \Gamma(f^*\Omega_X) \to S^{n+1} \xrightarrow{\psi} S.$$

The last term is S because any invertible sheaf on $W = \mathbf{A}^n - \{0\}$ is isomorphic to $\mathcal{O}_W$. This follows from the exact sequence (II, 6.5)

$$0 \to \operatorname{Pic} \mathbf{A}^n \xrightarrow{\sim} \operatorname{Pic} W \to 0$$

and the fact that $\operatorname{Pic} \mathbf{A}^n = 0$. Take generators $e_0, \dots, e_n$ of S^{n+1}. Then ψ is the map $e_i \mapsto x_i$, i.e. the multiplication by x map.

To finish the proof we need to know that if $\mathcal{F}$ is a coherent sheaf on $\mathbf{P}^n$, then $\Gamma(W, f^*\mathcal{F})\tilde{} = \mathcal{F}$. This assertion is completely natural but *doesn't* carry the force of proof, i.e., Hartshorne gave no proof. Taking global sections and applying this to the above sequence yields the exact sequence of sheaves on X

$$0 \to \Omega_{X/k} \to \Omega_X(-1)^{n+1} \to \mathcal{O}_X \to 0$$

which is just what we want. $\square$

Proposition 29.6. *If X is a nonsingular variety over a field k, then Ω_X is locally free of rank $n = \dim X$.*

The proof can be found in the book.

Example 29.7. Let C be a nonsingular curve in $\mathbf{P}^2_k$ defined by an equation f of degree d. Then $\Omega_{C/k}$ is a locally free sheaf of rank 1 so it corresponds to a divisor. Which divisor class will $\Omega_{C/k}$ correspond to? Let $\mathcal{I}$ be the ideal sheaf of $C \subset \mathbf{P}^2$. Then we have an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbf{P}^2} \otimes \mathcal{O}_C \to \Omega_{C/k} \to 0.$$

The left map is injective since $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_C(-d)$ so $\mathcal{I}/\mathcal{I}^2$ is locally free of rank 1. [[Why is $\mathcal{I}/\mathcal{I}^2 \cong \mathcal{O}_C(-d)$?]] Taking the second exterior power gives an isomorphism

$$\Lambda^2(\Omega_{\mathbf{P}^2} \otimes \mathcal{O}_C) \cong (\mathcal{I}/\mathcal{I}^2) \otimes \Omega_{C/k}.$$

This is a fact from the general theory of locally free sheaves. We have an exact sequence

$$0 \to \Omega_{\mathbf{P}^2} \to \mathcal{O}(-1)^3 \to \mathcal{O} \to 0$$

thus

$$\Lambda^3(\mathcal{O}(-1)^3) \cong \Lambda^2\Omega_{\mathbf{P}^2} \otimes \Lambda^1\mathcal{O}_{\mathbf{P}^2}$$

so $\mathcal{O}(-3) \cong \Lambda^2\Omega_{\mathbf{P}^2}$. Thus

$$\mathcal{O}(-3) \cong \mathcal{I}/\mathcal{I}^2 \otimes \Omega_{C/k} \cong \mathcal{O}_C(-d) \otimes \Omega_{C/k}.$$

Tensoring with $\mathcal{O}(3)$ gives $\mathcal{O}_C \cong \mathcal{O}_C(3-d) \otimes \Omega_{C/k}$ so $\Omega_{C/k} \cong \mathcal{O}_C(d-3)$.

If C is a cubic then

$$\Omega_{C/k} = \mathcal{O}_C(3-3) = \mathcal{O}_C(0) = \mathcal{O}_C.$$

Furthermore

$$\Omega_{\mathbf{P}^1} = \mathcal{O}(-2) \not\cong \mathcal{O}(0) = \Omega_{C/k}$$

so a nonsingular plane cubic is not rational.

Proposition 29.8. *Suppose $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of locally free sheaves of ranks r', r, and r''. Then*

$$\Lambda^r\mathcal{E} \cong \Lambda^{r'}\mathcal{E}' \otimes \Lambda^{r''}\mathcal{E}''.$$

30 Sheaf of Differentials and Canonical Divisor

Theorem 30.1. *Let X be a nonsingular projective variety of dimension n. Then $\Omega^n_{X/k} \cong \omega_X$ where ω_X is the dualizing sheaf on X. Furthermore, $\Omega_{X/k}$ is locally free of rank n and so $\Omega^n_{X/k}$ is locally free of rank 1. Thus $\Omega^n_{X/k}$ is an invertible sheaf on X.*

Note that $\Omega^n_{X/k}$ is an (abusive) shorthand notation for $\Lambda^n\Omega_{X/k}$ and that $\Omega^n_{X/k}$ is *not* the direct sum of n copies of $\Omega_{X/k}$.

Recall the construction of the dualizing sheaf ω_X. Let $f : X \to \mathbf{P}^n$ be a finite morphism. Let $\omega_{\mathbf{P}^n} = \mathcal{O}_{\mathbf{P}^n}(-1)$. Then

$$\omega_X = f^!\mathcal{O}_{\mathbf{P}^n}(-1) = f^!\omega_{\mathbf{P}^n} = \mathcal{H}\mathrm{om}(f_*\mathcal{O}_X, \omega_{\mathbf{P}^n})\tilde{}.$$

Since X is Cohen-Macaulay, $f_*\mathcal{O}_X$ is locally free so $f^!\omega_{\mathbf{P}^n}$ is locally free of rank 1.

First we show that the theorem holds when $X = \mathbf{P}^n$. From last time we have an exact sequence

$$0 \to \Omega_{\mathbf{P}^n} \to \mathcal{O}(-1)^{n+1} \to \mathcal{O} \to 0$$

so taking highest exterior powers gives an isomorphism

$$\Lambda^{n+1}(\mathcal{O}(-1)^{n+1}) \cong \Lambda^n\Omega_{\mathbf{P}^n} \otimes \Lambda^1\mathcal{O} \cong \Lambda^n\Omega_{\mathbf{P}^n}.$$

For the last isomorphism we used the fact that $\Lambda^1\mathcal{O} \xrightarrow{\sim} \mathcal{O}$. But since the highest exterior power changes a direct sum into a tensor product

$$\Lambda^{n+1}(\mathcal{O}(-1)^{n+1}) = \mathcal{O}(-1)^{\otimes n+1} = \mathcal{O}(-n-1) = \omega_{\mathbf{P}^n}.$$

so Combining these shows that $\Lambda^n\Omega_{\mathbf{P}^n} = \omega_{\mathbf{P}^n}$.

Now suppose X is an arbitrary nonsingular projective variety of dimension n. As we have done before let $f : X \to \mathbf{P}^n$ be a finite morphism (do this using suitable linear projections). To prove the theorem it is enough to show that $f^!(\Omega^n_{\mathbf{P}^n/k}) \cong \Omega^n_{X/k}$ since $f^!(\Omega^n_{\mathbf{P}^n/k}) = f^!(\omega_{\mathbf{P}^n}) = \omega_X$. This is just a statement about differentials.

More generally suppose $f : X \to Y$ is a finite map and that both X and Y are nonsingular projective varieties of dimension n. Then we have the duality

$$\mathrm{Hom}_X(\mathcal{F}, f^!\mathcal{G}) = \mathrm{Hom}_Y(f_*\mathcal{F}, \mathcal{G}).$$

Thus to give a map $\varphi : \Omega^n_X \to f^!\Omega^n_Y$ is equivalent to giving a map $\overline{\varphi} : f_*\Omega^n_X \to \Omega^n_Y$.

It is an *open problem* to find a natural yet elementary way to define a map $f_*\Omega^n_X \to \Omega^n_Y$ which corresponds to an isomorphism $\Omega^n_X \to f^!\Omega^n_Y$. Because the map sends differentials "above" to differentials "below" it should be called a "trace" map. In Hartshorne's book the existence of such a map is proved by embedding X in some large $\mathbf{P}^N$, showing that $\Omega_X = \mathcal{E}\mathrm{xt}^{N-n}(\mathcal{O}_X, \omega_{\mathbf{P}^n})$ and then applying the "fundamental local isomorphism". The approach is certainly not elementary because it involves the higher $\mathcal{E}\mathrm{xt}$ groups.

Continue to assume $f : X \to Y$ is a finite morphism of nonsingular varieties of dimension n. Assume f is separable so that the field extension $K(X)/K(Y)$ is finite and separable. A theorem proved last time gives an exact sequence

$$f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0.$$

Since X is nonsingular of dimension n, Ω_X is locally free of rank n. Similarly Ω_Y is locally free of rank n so $f^*\Omega_Y$ is locally free of rank n. (Locally this is just the fact that $A^n \otimes_A B \cong B^n$.) Since localization commutes with taking the module of differentials the stalk at the generic point of $\Omega_{X/Y}$ is $\Omega_{K(X)/K(Y)}$. This is 0 since $K(X)/K(Y)$ is finite separable, thus $\Omega_{X/Y}$ is a torsion sheaf. By general facts about free modules this implies the map $f^*\Omega_Y \to \Omega_X$ is injective, i.e., the sequence

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

is exact.

We pause to consider the simplest illustrative example, namely the case when X is a parabola and Y is the line.

Example 30.2. Let $A = k[x]$ and let $B = k[x,y]/(x - y^2) \cong k[y]$ where k is a field of characteristic not equal to 2. Let $X = \operatorname{Spec} B$ and let $Y = \operatorname{Spec} A$. Let $f : X \to Y$ be the morphism induced by the inclusion $A \hookrightarrow B$ (thus $x \mapsto y^2$). Since $B \cong k[y]$ it follows that $\Omega_X = B dy$, the free B-module generated by dy. Similarly $\Omega_Y = A dx$. The exact sequence above becomes

$$\begin{array}{ccccccc} 0 & \to & \Omega_Y \otimes_A B & \to & \Omega_X & \to & \Omega_{X/Y} & \to & 0 \\ & & \| & & \| & & \wr\| & & \\ & & B dx & \to & B dy & \to & B dy/B(2y dy) & & \end{array}$$

The point is that $\Omega_{X/Y} = (k[y]/(2y))dy$ is a torsion sheaf supported on the ramification locus of the map $f : X \to Y$. (The only ramification point is above 0.) Note that $\Omega_{X/Y}$ is the quotient of Ω_X by the submodule generated by the image of dx in $\Omega_X = B dy$. The image of dx is $2y dy$.

More generally we define the ramification divisor as follows.

Definition 30.3. Let $f : X \to Y$ be a finite separable morphism of nonsingular varieties of dimension n. Then the **ramification divisor** of X/Y is

$$R = \sum_{\zeta \in Z \subset X} \mathrm{length}_{\mathcal{O}_\zeta}((\Omega_{X/Y})_\zeta) \cdot Z$$

where the sum is taken over all closed irreducible subsets $Z \subset X$ of codimension 1 and ζ is the generic point of Z.

Since the sequence

$$0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$$

is exact it will follow that

$$\Omega_X^n \cong f^*\Omega_Y^n \otimes \mathcal{L}(R)$$

where R is the ramification divisor of X/Y and $\mathcal{L}(R)$ denotes the corresponding invertible sheaf. [[This is some linear algebra over modules.]]

The next part of the argument is to study $f^!$. As usual let $f : X \to Y$ be a finite morphism of nonsingular varieties of dimension n and assume furthermore that $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module. Define a trace map $\mathrm{Tr} : f_*\mathcal{O}_X \to \mathcal{O}_Y$ locally as follows. Let $\operatorname{Spec} A$ be an open affine subset of Y and let $\operatorname{Spec} B = f^{-1}(\operatorname{Spec} A)$. Then B is a free A-module of rank $d = \deg f$. Choose a basis $e_1, \ldots, e_d$ for B/A. Let $b \in B$, and suppose $be_i = \sum_j a_{ij} e_j$. Define $\mathrm{Tr}(b) = \sum_i a_{ii}$. Let $\overline{\mathrm{Tr}} \in \mathrm{Hom}_X(\mathcal{O}_X, f^!\mathcal{O}_Y)$ correspond to $\mathrm{Tr} \in \mathrm{Hom}_Y(f_*\mathcal{O}_X, \mathcal{O}_Y)$ under the isomorphism between these Hom groups.

Claim. $f^!\mathcal{O}_Y \cong \mathcal{L}(R)$.

Once we have proved the claim, the theorem will follow. To see this tensor both sides by $f^*\Omega^n_Y$. Then

$$f^!\Omega^n_Y = f^!\mathcal{O}_Y \otimes f^*\Omega^n_Y = \mathcal{L}(R) \otimes f^*\Omega^n_Y = \Omega^n_{X/Y}.$$

We look what happens locally. Let $\operatorname{Spec} A \subset Y$ and $\operatorname{Spec} B = f^{-1}(\operatorname{Spec} A) \subset X$. We want to show that $f^!(\mathcal{O}_Y) = \mathcal{L}(R)$. Since $f^!\mathcal{O}_Y = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y)\tilde{}$ we look at $B^* = \mathrm{Hom}_A(B, A)$. First consider the special case of the parabola investigated above. Then B has a basis $1, y$ over A and B^* is spanned by e_0 and e_1 where $e_0(1) = 1$, $e_0(y) = 0$, and $e_1(1) = 0$, $e_1(y) = 1$. Thus $ye_0(1) = e_0(y) = 0$, $ye_0(y) = e_0(y^2) = e_0(x) = xe_0(1) = x$, and $ye_1(1) = e_1(y) = 1$, $ye_1(y) = e_1(y^2) = e_1(x) = xe_1(1) = 0$. Therefore $ye_1 = e_0$ so e_1 generates B^* over A. The trace $Tr : B \to A$ is an element of B^*. We determine it. We see that $Tr(1) = 2$ and $Tr(y) = 0$ since

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } y \leftrightarrow \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}$$

Thus $\mathrm{Tr} = 2e_0 = 2ye_1$ since $ye_1 = e_0$ as shown above. We have shown that the map

$$\overline{\mathrm{Tr}} : B \to B^* : 1 \mapsto \mathrm{Tr}$$

has image generated by $2ye_1$. More generally suppose $B = A[y]/(f(y))$ where $f(y) = y^d + a_1 y^{d-1} + \cdots + a_d$ is a monic polynomial of degree d. Then B has A-basis $1, y, y^2, \ldots, y^{d-1}$ and B^* is spanned by $e_0, e_1, \ldots, e_{d-1}$. By a similar argument to the one above, we can show that $\mathrm{Tr} = f'(y)e_{d-1}$. Thus locally $f^!\mathcal{O}_Y$ is a free $\mathcal{O}_X$-module of rank 1 generated by $f'(y)e_{d-1}$. But this is really what we defined $\mathcal{L}(R)$ to be (where this derivative is zero is where there is ramification). Thus $f^!\mathcal{O}_Y = \mathcal{L}(R)$ and we are done.

Curves

31 Definitions

Definition 31.1. A **curve** is a connected nonsingular projective 1-dimensional scheme over an algebraically closed field k.

We have proved that if C_1 and C_2 are curves then $C_1 \cong C_2$ iff C_1 is birational to C_2 (i.e., they have isomorphic open subsets) iff $K(C_1) \cong K(C_2)$. Here $K(C_1) = \mathcal{O}_{C,\zeta}$ where ζ is the generic point of C_1. If any of the hypothesis nonsingular, connected, projective, or one-dimensional is removed then these equivalences can fail to hold.

32 Genus

Let $X \hookrightarrow \mathbf{P}^n$ be an embedded projective variety. Let $P_X(n)$ be its Hilbert polynomial. The **arithmetic genus** of X is the quantity p_a defined by the equation $P_X(0) = 1 + (-1)^{\dim X} p_a$. Thus if C is a curve then $p_a = 1 - P_X(0)$. For a nonsingular projective curve we also define the quantity $g = h^1(\mathcal{O}_C)$ and call it the **genus**. The **geometric genus** $p_g = h^0(\omega_X)$ is a third notion of genus for a nonsingular projective variety X. For curves all types of genus coincide.

Theorem 32.1. *If C is a curve then*

$$p_a = g = p_g.$$

Proof. $P_C(n) = \sum (-1)^i h^i(\mathcal{O}_X(n))$ so $P_C(0) = \sum(-1)^i h^i(\mathcal{O}_X)$. But C has dimension 1 so by Grothendieck vanishing

$$1 - p_a = P_C(0) = h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = 1 - h^1(\mathcal{O}_C) = 1 - g$$

so $p_a = g$.

Serre duality says that if X is nonsingular of dimension n then $\mathrm{Ext}^i_X(\mathcal{F}, \omega_X)$ is linearly dual to $H^{n-i}(X, \mathcal{F})$. Thus when $X = C$, $\mathcal{F} = \mathcal{O}_C$, $i = 0$, and $n = 1$ we see that

$$H^0(\omega_C) = \mathrm{Ext}^0_C(\mathcal{O}_C, \omega_C)$$

is dual to $H^1(\mathcal{O}_C)$. Thus $p_g = h^1(\mathcal{O}_C) = h^0(\omega_C) = g$. $\square$

Thus we may speak of *the* genus of a curve. For more general varieties the concepts diverge.

The classification problem is to describe all curves up to isomorphism. The set of curves is a disjoint union $\mathcal{C} = \coprod_{g \geq 0} \mathcal{C}_g$ where $\mathcal{C}_g$ consists of all curves of genus g.

Theorem 32.2. *Any curve of genus 0 is isomorphic to $\mathbf{P}^1$.*

Proof. Let C be a curve of genus 0 and let $P \in C$ be a closed point. There is an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{L}(P) \to \kappa(P) \to 0.$$

AS a divisor P corresponds to the invertible sheaf $\mathcal{L}(P)$. Let $s \in \Gamma(\mathcal{L}(P))$ be a global section which generates $\mathcal{L}(P)$ as an $\mathcal{O}_C$-module. Thus s has a pole of order 1 at P and no other poles. Then s defines a morphism

$$\mathcal{O}_C \to \mathcal{L}(P) : 1 \mapsto s$$

which has cokernel $\kappa(P)$. Taking cohomology yields

$$0 \to H^0(\mathcal{O}_C) \to H^0(\mathcal{L}(P)) \to H^0(\kappa(P)) \to H^1(\mathcal{O}_C) = 0.$$

Since $H^0(\mathcal{O}_C) = k$ and $H^0(\kappa(P)) = k$ it follows that $H^0(\mathcal{L}(P)) = k \oplus k$. View $\mathcal{O}_C \subset \mathcal{L}(P) \subset K$ where $K = K(X)$ is the constant function field sheaf. Then

$$H^0(\mathcal{O}_C) \hookrightarrow H^0(\mathcal{L}(P)) \hookrightarrow K.$$

Since $\dim H^0(\mathcal{L}(P)) > \dim H^0(\mathcal{O}_C)$, there is $f \in H^0(\mathcal{L}(P)) - H^0(\mathcal{O}_C)$ and f is non constant. Thus $f \in K = K(X)$ and $f \notin k$. So f defines a morphism $C \to \mathbf{P}^1$ as follows. If $Q \in C$ and $f \in \mathcal{O}_Q$ send Q to $\overline{f} \in \mathcal{O}_Q/\mathbf{m}_Q = k \subset \mathbf{P}^1_k$. If $Q \notin C$ so that f has a pole at Q, send Q to $\infty \in \mathbf{P}^1_k$. Since f lies in $H^0(\mathcal{L}(P))$ which has dimension 1 over $k = H^0(\mathcal{O}_C)$, $v_P(f) = -1$. Thus ∞ does not ramify so the degree of the morphism defined by f is the number of points lying over ∞ which is 1. Thus $\mathbf{P}^1_k \cong C$. $\square$

Remark 32.3. This result is false if k is not algebraically closed. But it is true if C has a rational point, i.e., a point $P \in C$ so that $\kappa(P) = \mathcal{O}_P/\mathbf{m}_P = k$.

Example 32.4. Let $k = \mathbf{R}$ and let C be the curve $x^2 + y^2 + z^2 = 0$ on $\mathbf{P}^2_{\mathbf{R}}$. C is nonsingular of degree 2 so $g = 0$. But C is not isomorphic over $\mathbf{R}$ to $\mathbf{P}^1_{\mathbf{R}}$ since C has no rational points whereas $\mathbf{P}^1_{\mathbf{R}}$ does. Let $P = (1, i, 0) \in C_{\mathbf{C}}$ and $\overline{P} = (1, -i, 0) \in C_{\mathbf{C}}$. Let $P^* \in C_{\mathbf{R}}$ be the Galois conjugacy class $\{P, \overline{P}\}$. Then $\kappa(P^*) = C$ is of degree 2 over $\mathbf{R}$.

Exercise 32.5. Over $\mathbf{R}$ an curve of genus 0 not isomorphic to $\mathbf{P}^1_{\mathbf{R}}$ is isomorphic to

$$C : x^2 + y^2 + z^2 = 0$$

.

[[If the curve is planar this follows from diagonalizability of quadratic forms.]]

33 Riemann-Roch Theorem

The Riemann-Roch theorem is the cornerstone of all of curve theory.

Theorem 33.1 (Riemann-Roch). *Let C be a curve and $D = \sum_{i=1}^{t} n_i P_i$ be a divisor on C. Then*

$$h^0(\mathcal{L}(D)) - h^1(\mathcal{L}(D)) = \deg D + 1 - g$$

where $\deg D = \sum_{i=1}^{t} n_i$.

Proof. When $D = 0$ the assertion is that

$$h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = 1 - g$$

which is easily checked. Next suppose D is any divisor and P any point. Compare D and $D + P$ as follows. The statement of the theorem for D is

$$h^0(\mathcal{L}(D)) - h^1(\mathcal{L}(D)) = \deg D + 1 - g$$

and for $D + P$ the theorem is

$$h^0(\mathcal{L}(D + P)) - h^1(\mathcal{L}(D + P)) = \deg D + 1 + 1 - g.$$

Use the exact sequence

$$0 \to \mathcal{L}(D) \to \mathcal{L}(D + P) \to \kappa(P) \to 0$$

to compute the Euler characteristic $\chi = h^0 - h^1$ of $\mathcal{L}(D + P)$. This yields

$$\chi(\mathcal{L}(D + P)) = \chi(\mathcal{L}(D)) + \chi(\kappa(P))$$

so

$$h^0(\mathcal{L}(D + P)) - h^1(\mathcal{L}(D + P)) = h^0(\mathcal{L}(D)) - h^1(\mathcal{L}(D)) + 1.$$

The theorem is thus true for $D+P$ iff it is true for D. Since the theorem is true for $D = 0$ and we can obtain any divisor by adding or subtracting points starting with $D = 0$ the theorem follows. □

34 Serre Duality

By Serre Duality $H^1(\mathcal{L}(D))$ is dual to

$$\begin{aligned}\operatorname{Ext}^0(\mathcal{L}(D), \omega_C) &= \operatorname{Ext}^0(\mathcal{O}_C, \omega_C \otimes \mathcal{L}(D)^\vee) \\ &= H^0(\omega_C \otimes \mathcal{L}(-D)) = H^0(\mathcal{L}(K-D))\end{aligned}$$

Thus $h^1(\mathcal{L}(D)) = h^0(\mathcal{L}(K-D))$ where K is some divisor so that $\mathcal{L}(K) = \omega$. K is often called the canonical divisor. Thus we can restate the Riemann-Roch theorem as

$$h^0(\mathcal{L}(D)) - h^0(\mathcal{L}(K-D)) = \deg D + 1 - g.$$

Sometimes one abbreviates $h^0(\mathcal{L}(D))$ as $\ell(D)$ and then Riemann-Roch becomes

$$\mathcal{L}(D) - \mathcal{L}(K-D) = \deg D + 1 - g.$$

Corollary 34.1. $\deg K = 2g - 2$.

Proof. By Riemann-Roch,

$$h^0(\mathcal{L}(K)) - h^0(\mathcal{L}(0)) = \deg K + 1 - g$$

but $h^0(\mathcal{L}(K)) = g$ and $h^0(L(0)) = 1$. Thus $\deg K = 2g - 2$. □

Homework II, Exercise 8.4; III, Exercise 6.8, 7.1, 7.3

35 A Bird's Eye View of Curves

$$3g \quad - \quad 3 \quad + \quad g \quad + \quad 4(d+1-g-4) \quad + \quad 15 \quad = \quad 4d$$

1. $3g-3$ is the dimension of the moduli space $\mathcal{M}_g$ of curves of genus g.
2. g is the dimension of $\operatorname{Pic}^d C$.
3. $4(d+1-g-4)$ is the number of ways to choose a linear system W in $W^0(\mathcal{L})$.
4. 15 is the dimension of $\operatorname{Aut} \mathbf{P}^3 \cong \mathrm{PGL}(4,k)$.
5. $4d$ is the dimension of the Hilbert scheme $H^0_{d,g}$ of nonsingular curves of genus g and degree d.

36 Moduli Space

As a set $\mathcal{M}_g$ is the set of curves of genus g modulo isomorphism. $\mathcal{M}_g$ can be made into a variety in a natural way. As a variety $\mathcal{M}_g$ is irreducible and

$$\dim \mathcal{M}_g = \begin{cases} 0 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 3g-3 & \text{if } g \geq 2 \end{cases}$$

$\mathcal{M}_g$ is not a projective variety, but its closure $\overline{\mathcal{M}}_g$ is. The points of $\overline{\mathcal{M}}_g$ not in $\mathcal{M}_g$ are called stable curves.

37 Embeddings in Projective Space

38 Elementary Curve Theory

38.1 Definitions

Let C be a curve thus C is a nonsingular connected projective variety of dimension 1 over an algebraically closed field k. Let g be the genus of C. A divisor D is a sum $\sum_{i=1}^k n_i P_i$ where $n_i \in \mathbf{Z}$ and P_i is a closed point of C. Let $\deg D = \sum_{i=1}^k n_i$. A divisor D corresponds to an invertible sheaf $\mathcal{L}(D) = \mathcal{I}_D^\vee$ where $\mathcal{I}_D$ is the ideal sheaf of D and $\mathcal{I}_D^\vee = \mathrm{Hom}_C(\mathcal{I}_D, \mathcal{O}_C)$ is the dual of $\mathcal{I}_D$. Note that $\mathcal{L}(nD) = \mathcal{L}(D)^{\otimes n}$ since $\otimes$ and $\vee$ commute.

Notation: $\mathcal{O}(D) := \mathcal{L}(D)$, $\mathcal{M}(D) = \mathcal{M} \otimes \mathcal{L}(D)$.

We say a divisor $\sum n_i D_i$ is effective if all $n_i \geq 0$. Let $|D|$ denote the complete linear system associated to D. Thus

$$|D| = \{D' : D' \text{ is effective and } D' \sim D\}.$$

Here $\sim$ denotes *linear equivalence*. Two divisors D and D' are linearly equivalent if there exists $f \in K = K(C)$ (the function field of C) so that $D - D' = (f)$ where $(f) = \sum_{P \in C \text{ closed}} v_P(f) P$.

Since the condition $D + (f) \geq 0$ is exactly the condition that $f \in H^0(C, \mathcal{L}(D))$, it follows that there is a bijection

$$\begin{aligned} |D| &\xrightarrow{\sim} H^0(C, \mathcal{L}(D))/k^* \\ D' = D + (f) &\mapsto f. \end{aligned}$$

Two functions f, g which differ by an element of k^* give the same divisor. Note that $|D|$ may be empty.

If $s \in H^0(\mathcal{L})$ then there is an injection $0 \to \mathcal{O} \xrightarrow{s} \mathcal{L}(D)$ given by multiplication by s. Dualizing we obtain

$$\mathcal{L}(-D) = \mathcal{I}_D = \mathcal{L}^\vee \xrightarrow{s^\vee} \mathcal{O} \to \mathcal{O}_D \to 0.$$

Thus $D \neq 0$ is effective iff D corresponds to a closed subscheme defined locally by $(t_{P_i}^{n_i} \subset \mathcal{O}_{P_i}$ where $t_{P_i} \in \mathbf{m}_{P_i} \subset \mathcal{O}_{P_i}$ is a uniformizing parameter at $\mathbf{m}_{P_i}$.

$|D|$ can be regarded as a projective space and as such

$$\dim |D| = h^0(\ell(D)) - 1 = \ell(D) - 1,$$

where $\ell(D) = h^0(\mathcal{L}(D))$. We can generalize the notion of a complete linear system by considering linear subspaces of $|D|$. A *(general) linear system* is a linear subspace $\mathcal{D} \subset |D|$. Thus $\mathcal{D}$ corresponds to a vector subspace $W \subset H^0(C, \mathcal{L}(D))$.

38.2 Maps to Projective Space

39 Low Genus Projective Embeddings

Let C be a curve of genus g, let K be the canonical divisor (which corresponds to the invertible sheaf $\Omega = \omega$ of differentials) and let D be a divisor.

Theorem 39.1 (Riemann-Roch). $\ell(D) - \ell(K - D) = \deg(D) + 1 - g$

Proposition 39.2. *A divisor D is ample iff* $\deg D > 0$.

Proposition 39.3. *If* $\deg D \geq 2g + 1$ *then D is very ample.*

39.1 Genus 0 curves

Suppose C is a curve of genus $g = 0$. If $\deg D \geq 1$ then D is very ample. Suppose $D = P$ is a point. Then D gives rise (after a choice of basis for the corresponding invertible sheaf) to an embedding $C \hookrightarrow \mathbf{P}^n$ where $n = \ell(D) - 1 = 1$. When $D = 2P$ we obtain the 2-uple embedding $C \hookrightarrow \mathbf{P}^2$ which is a conic since $g = 0$. When $D = 3P$, $C \hookrightarrow \mathbf{P}^2$ is the twisted cubic. More generally $D = dP$ gives the rational normal curve $C \hookrightarrow \mathbf{P}^d$ of degree d (which is in fact projectively normal).

39.2 Genus 1 curves

If C is a curve of genus $g = 1$ then $\deg D \geq 2g + 1 = 3$ iff D is very ample. Thus the converse to proposition 2 holds when $g = 1$. Suppose D is a very ample divisor of degree 2. (For degree 1 or 0 a similar argument works.) Then D gives rise to an embedding $C \hookrightarrow \mathbf{P}^1$ since $\ell(D) - 1 = 2 - 1 = 1$ which is absurd since C has genus 1 but $\mathbf{P}^1$ has genus 0. To see that $\ell(D) = 2$ apply Riemann-Roch to obtain $\ell(D) - \ell(K - D) = 2 + 1 - 1 = 2$. Then since $\deg K = 2g - 2 = 0$ we see that $\deg(K - D) < 0$ and hence $\ell(K - D) = 0$ so $\ell(D) = 2$ as desired.

Suppose D is a divisor of degree 3 on a genus 1 curve C. Then D gives rise to an embedding $C \hookrightarrow \mathbf{P}^2$ since $\ell(D) - 1 = (\deg D + 1 - g) - 1 = 3 - 1 = 2$. Thus any curve of genus 1 can be embedded as a nonsingular cubic curve in $\mathbf{P}^2$. This embedding also allows us to show that $K \sim 0$. We showed before that if $C \subset \mathbf{P}^2$ is of degree d then $\omega_C \cong \mathcal{O}_C(d - 3)$. Thus choosing an embedding arising from a divisor of degree 3 as above we see that $\omega_C \cong \mathcal{O}_C(0)$, thus the canonical sheaf corresponds to the trivial divisor class so $K \sim 0$. Alternatively, we can prove this by using Riemann-Roch to see that

$$\ell(K) = \ell(0) + \deg(K) + 1 - g = 1 + 0 + 1 - 1 = 1$$

and thus K is linearly equivalent to an effective divisor of degree $2g - 2 = 0$.

39.3 Moduli Space

The moduli spaces of curves of low genus are

$$\mathcal{M}_0 = \{ \text{ curves of genus } 0 \ \} = \{\mathbf{P}^1\}$$

$$\mathcal{M}_1 = \{ \text{ curves of genus } 1 \ \} = \mathbf{A}^1$$

A curve C of genus $g = 1$ can be given by a degree 3 embedding $C \hookrightarrow \mathbf{P}^2$. We will show [[in nice characteristic only?]] that two embedded curves $C_1 \hookrightarrow \mathbf{P}^2$ and $C_2 \hookrightarrow \mathbf{P}^2$ are isomorphic as abstract curves iff there is an automorphism in $\operatorname{Aut} \mathbf{P}^2 = \mathrm{PGL}(3)$ sending C_1 to C_2. Note that $\operatorname{Aut} \mathbf{P}^2$ is a $3^2 - 1 = 8$ dimensional family. A degree 3 curve $C \hookrightarrow \mathbf{P}^2$ is given by a degree 3 polynomial

$$F = a_0 x^3 + \cdots \in H^0(\mathcal{O}_{\mathbf{P}^2}(3)).$$

Since $\dim H^0(\mathcal{O}_{\mathbf{P}^2}(3)) = 10$ we obtain a $10 - 1 = 9$ dimensional projective space of such curves (including the singular ones). The nonsingular ones form an open subset $\Delta \neq 0$. The moduli space $\mathcal{M}_g$ has dimension $9 - 8 = 1$. [[Not clear.]]

How can degree 3 embedded curves $C_1, C_2 \subset \mathbf{P}^2$ be isomorphic? We generalize to arbitrary degree and ask the following question.

Question 39.4. *Suppose $C_1, C_2 \subset \mathbf{P}^2$ are both nonsingular curves of degree d and suppose $C_1 \cong C_2$ as abstract curves. Does is necessarily follow that there is an automorphism g of $\mathbf{P}^2$ sending C_1 to C_2, i.e., such that $g(C_1) = C_2$?*

Suppose $d = 1$. Then C_1 and C_2 are both lines in $\mathbf{P}^2$ so the answer is yes.

Suppose $d = 2$. Then C_1 and C_2 are both conics. If k is algebraically closed and $\operatorname{char} k \neq 2$ the defining equations C_1 and C_2 can be transformed into $x^2 + y^2 + z^2 = 0$ by an automorphism of $\mathbf{P}^2$ (by "completing the square"). Thus in this case the answer to the question is yes. When $\operatorname{char} k = 2$ the answer is no. [[give an easy counterexample here.]]

Suppose $d = 3$. Thus C_1 and C_2 are both cubic curves in $\mathbf{P}^2$ and $C_1 \cong C_2$ as abstract curves. Equivalently we are given an abstract curve C and two embeddings

$$\varphi_1 : C \hookrightarrow \mathbf{P}^2$$

$$\varphi_2 : C \hookrightarrow \mathbf{P}^2$$

The question is then: does there exist an automorphism g of $\mathbf{P}^2$ such that $g(\varphi_1(C)) = \varphi_2(C)$? The embedding data giving φ_1 is a divisor D along with a basis of global sections $s_0, s_1, s_2 \in H^0(\mathcal{O}_C(D))$. An automorphism of $\mathbf{P}^2$ induces a map on C which preserves D but changes the basis s_0, s_1, s_2. Suppose φ_1 is given by D_1 and global sections s_0, s_1, s_2, and that φ_2 is given by D_2 and t_0, t_1, t_2. The automorphism g induces a map

$$g' : C \cong \varphi_1(C) \xrightarrow{g} \varphi_2(C) \cong C.$$

Then $g'(D_1) = D_2$ and $g' : s_0, s_1, s_2 \mapsto t_0, t_1, t_2$.

This generalizes so that in degree $d \geq 3$ a necessary condition for a yes answer to the above question is that for any two divisors D_1 and D_2 of degree d on C there is $g \in \operatorname{Aut} \mathbf{P}^2$ such that $g'(D_1) = D_2$.

40 Curves of Genus 3

Today we will study curves of genus 3.

Example 40.1. A plane curve $C \subset \mathbf{P}^2$ of degree $d = 4$ has genus $\frac{1}{2}(d-1)(d-2) = 3$.

Example 40.2. A curve C on the quadric surface Q in $\mathbf{P}^3$ of type (2,4) has degree 6 and genus 3.

These two examples are *qualitatively* different. The curves in the first example are "canonical" whereas the curves in the second class are "hyperelliptic".

We consider the first example in more detail. Let $C \subset \mathbf{P}^2$ be a genus 3 plane curve (so C has degree $d = 4$). Then

$$\omega_C = \mathcal{O}_C(d-3) = \mathcal{O}_C(1)$$

so ω_C is very ample. Thus the canonical embedding arising from the canonical divisor is exactly the given embedding $C \hookrightarrow \mathbf{P}^2$.

Next we consider example 2 in more detail. Let C be a genus three curve of type (2,4) on the quadric surface $Q \cong \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$. The projections $p_1, p_2 : Q \to \mathbf{P}^1$ give rise to a degree 2 and a degree 4 morphism of C to $\mathbf{P}^1$. Thus there exists a 2-to-1 morphism $f : C \to \mathbf{P}^1$. f corresponds to a base point free linear system on C of degree 2 and dimension 1. This linear system in turn corresponds to an effective divisor D of degree 2 with $\ell(D) = 2$ so $|D| = 1$. The existence of such a divisor means there exists P, Q such that $|P + Q|$ has dimension 1.

Definition 40.3. A curve C is *hyperelliptic* if $g \geq 2$ and there is a base point free linear system of degree 2 and dimension 1.

It is classical notation that a base point free linear system of degree d and dimension r is called a g^r_d. To say that a curve is hyperelliptic is to say that it has a g^1_2.

- If $g = 0$ then there is always a g^2_2.
- If $g = 1$ any divisor of degree 2 gives a g^1_2 by Riemann-Roch. Indeed, if D has degree 2 then
$$\dim |D| - \dim |K - D| = 2 + 1 - g = 2$$
and $\deg(K - D) = -2$ so $\dim |D| - (-1) = 2$ and hence $\dim |D| \leq 1$.
- If $g = 2$ every curve is hyperelliptic since $|K|$ is a g^1_2. Indeed, applying Riemann-Roch we see that
$$\dim |K| = \dim |K| - \dim |0| = 2 - 1 = 1.$$

Lemma 40.4. *Let C be any curve and D any divisor of degree $d > 0$. Then* $\dim |D| \leq d$ *with equality iff C is a rational curve.*

Proof. This is (IV, Ex. 1.5) in Hartshorne. Although one might guess that this lemma follows from Riemann-Roch this is not the case. Riemann-Roch gives a different sort of relationship between the dimension and degree of a divisor. We induct on d.

First suppose $d = 1$. First note that
$$\dim |P| = \ell(P) - 1 = h^0(\mathcal{O}_C(P)) - 1.$$
There is an exact sequence
$$0 \to \mathcal{O}_C \to \mathcal{O}_C(P) \to k(P) \to 0.$$
Now $h^0(\mathcal{O}_C) = 1$ and $h^0(k(P)) = 1$ therefore $h^0(\mathcal{O}_C(P)) \leq 2$ so $\dim |P| \leq 1$. If $\dim |P| = 1$ then $|P|$ has no base points so we obtain a morphism $C \to \mathbf{P}^1$ of degree $\deg P = 1$ which must be an isomorphism so C is rational.

Next suppose $D = P_1 + \cdots + P_d$. Let $D' = P_1 + \cdots + P_{d-1}$. There is an exact sequence
$$0 \to \mathcal{O}_C(D') \to \mathcal{O}_C(D) \to k(P_d) \to 0.$$
Now $h^0(\mathcal{O}_C(D')) \leq d$ by induction and $h^0(k(P_d)) = 1$ so $h^0(\mathcal{O}_C(D)) \leq d + 1$, therefore $\dim |D| \leq d$ with equality iff $h^0(\mathcal{O}_C(D')) = d$. By induction $h^0(\mathcal{O}_C(D')) = d$ iff C is rational. □

Theorem 40.5. *Suppose C is a curve of genus $g \geq 2$. Then ω_C is very ample iff C is not hyperelliptic.*

Proof. Let K be the canonical divisor. Then by a previous result K is very ample iff for all points P, Q, $\dim |K - P - Q| = \dim |K| - 2$. By Riemann-Roch,
$$\dim |P + Q| - \dim |K - P - Q| = 2 + 1 - g = 3 - g.$$
Now K is very ample iff $\dim |K - P - Q| = \dim |K| - 2 = (g - 1) - 2 = g - 3$ so when K is very ample the above becomes
$$\dim |P + Q| - (g - 3) = 3 - g.$$
Thus K is very ample iff for all P and Q, $\dim |P + Q| = 0$. Thus K is not very ample iff there exists P and Q so that $\dim |P+Q| = 1$. But the latter condition occurs precisely when C is hyperelliptic. (We can exclude the case $\dim |P + Q| \geq 2$ by using the previous lemma and the fact that C is not rational.) □

Corollary 40.6. *If C is a curve of genus $g \geq 1$ then $|K|$ has no base points.*

Proof. If $g = 1$ then $K = 0$ so $|K| = \{0\}$ and we are done. If $g \geq 2$ then $|K|$ is base point free iff

$$\dim |K - P| = \dim |K| - 1$$

for all P. (If $|K|$ has a base point P then every effective divisor D linearly equivalent to K is such that $D - P$ is effective and linearly equivalent to $K - P$. If $|K|$ has no base points then the dimension of the space of effective divisors equivalent to $K - P$ must go down for ever P.) Since

$$\dim |P| = \dim |K - P| + 1 + 1 - g$$

and

$$\dim |K| = 2g - 2 + 1 - g = g - 1$$

we see that

$$\dim |K - P| = \dim |K| - 1 + \dim |P|$$

so $|K|$ is base point free iff $\dim |P| = 1$ for all P. But $\dim |P| \leq 1$ for every P and we have equality iff C is rational (i.e., g=0). Since C is not rational it follows that $|K|$ is base point free. □

Suppose given an abstract curve C of genus 3. Then C belongs to one of two disjoint classes. If the canonical sheaf ω_C is very ample then we obtain an embedding of C into $\mathbf{P}^{g-1} = \mathbf{P}^2$ as a nonsingular quartic curve. If ω_C is not very ample then C is hyperelliptic (since the map induced by ω_C is 2-to-1). Does every hyperelliptic curve arise as a curve of type (2,4) on $Q \subset \mathbf{P}^3$? Hartshorne claims to have three-fourths of a proof.

Lemma 40.7. *If C is any curve of genus 3 then there exists a very ample divisor of degree $6 = 2g$.*

Proof. Let D be a divisor of degree 6. We have shown that D is very ample iff

$$\dim |D - P - Q| = \dim |D| - 2$$

for all P and Q. Since $\deg K = 2g - 2 = 4$ Riemann-Roch asserts that

$$\dim |D| = \dim |K - D| + 6 + 1 - g = -1 + 6 + 1 - g = 6 - g = 3$$

$$\dim |D - P| = \dim |K - (D - P)| + 5 + 1 - g = 5 - g = 2$$

$$\dim |D - P - Q| = \dim |K - (D - P - Q)| + 4 + 1 - g.$$

Thus for D to be very ample it must be the case that

$$\dim |K - (D - P - Q)| = -1.$$

Since $\deg(K - (D - P - Q)) = 0$ it follows that $\dim |K - (D - P - Q)| = -1$ iff K is not linearly equivalent to $D - P - Q$. The assertion is thus reduced to showing that among all divisors of degree 6 the set with $D - P - Q \sim K$ is a proper closed subset.

[[I do not understand Hartshorne's proof of this. He says $D - P - Q \sim K$ iff $D \sim K + P + Q$. He then claims that the family of D of degree 6 is a 6 dimensional family and that the family of divisor $K + P + Q$ is a 5 dimensional family.]] □

What about the converse? If C is hyperelliptic must C then have to lie on a quadric surface?

41 Curves of Genus 4

Recall that curves of genus $g \geq 2$ split up into two disjoint classes.

(a) hyperelliptic

(b) ω_X is very ample

If $g = 3$ and C is of type $(2, 4)$ on $Q \subset \mathbf{P}^3$ then C is hyperelliptic. Also, if $g = 3$ then ω_X is very ample iff C is a degree 4 curve in $\mathbf{P}^2$. Any curve of genus $g = 3$ can be embedded as a curve of degree 6 in $\mathbf{P}^3$. We do not know whether any such curve can actually be put on a quadric surface. "This would make a great homework problem — I do not know the answer."

Next we consider curves of genus 4.

Example 41.1. Consider a type $(2, 5)$ curve C on $Q \subset \mathbf{P}^3$. Then C has degree $7 = 2 + 5$ and C is hyperelliptic (because of the degree 2 map coming from projection onto the first fact $p_1 : Q \to \mathbf{P}^1$). A type $(3, 3)$ curve on Q is also of genus 4. It is a degree 6 complete intersection of Q and a cubic surface. Curves of type $(3, 3)$ have at least two g_3^1's.

41.1 Aside: existence of g_d^1's in general

Assume for this discussion that g_d^1's are allowed to have base points. Call such g_d^1's trivial. Given a g_d^1 adding a point p trivially gives a (trivial) g_{d+1}^1.

Question 41.2. *Given any curve C what is the least d for which there exists a g_d^1?*

- $g = 0$ there is a g_1^1 coming from the embedding $\mathbf{P}^1 \hookrightarrow \mathbf{P}^1$,
- $g = 1$ there are infinitely many g_2^1's,
- $g = 2$ there is a g_2^1 namely ω_C,
- $g \geq 2]$ if there is a g_2^1 then C is hyperelliptic,
- $g > 2$ there exists nonhyperelliptic curves.

If a curve C has a g_3^1 it is called *trigonal.* If $g \leq 2$ then there exists a trivial g_3^1 (just add a point to a g_2^1). If $g = 3$ and C is hyperelliptic then it is trivial that there is a g_3^1. If C is not hyperelliptic it is not trivial. But projection through a point $P \in C$ gives a 3-to-1 map to $\mathbf{P}^1$. This corresponds to a g_3^1. Thus a curve of genus 3 has infinitely many g_3^1's (one for each point, coming from projection). If $g = 4$ and C is hyperelliptic then it is trivial that there is a g_3^1. In general given any genus 4 curve one can always construct a g_3^1. When $g \geq 5$ in general there will not be a g_3^1. This pattern repeats itself.

41.2 Classifying curves of genus 4

Start with an abstract curve C of genus 4. We do not deal with the case C hyperelliptic now. A related question is the following.

Question 41.3. *Does every hyperelliptic curve live on the quadric surface?*

We postpone this question or maybe put it on an upcoming homework assignment. [Everyone shudders.]

If C is not hyperelliptic then ω_C is very ample. Therefore we have the canonical embedding $C \hookrightarrow \mathbf{P}^{g-1} = \mathbf{P}^3$. The degree of the embedded curve is $\deg \omega_C = 2g-2 = 6$. Thus view C as a degree 6 genus 4 curve in $\mathbf{P}^3$. What does C lie on? There is an exact sequence

$$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbf{P}^3}(2)) \to H^0(\mathcal{O}_C(2)) \to \cdots$$

Since Riemann-Roch states that $\ell(D) = \deg D + 1 - g + h^1(\mathcal{O}(D))$ we see that $h^0(\mathcal{O}_C(2)) = 12+1-4+0 = 9$. [[This is not quite clear.]] Since $h^0(\mathcal{O}_{\mathbf{P}^3}(2)) = 10$ it follows that the map $H^0(\mathcal{O}_{\mathbf{P}^3}(2)) \to H^0(\mathcal{O}_C(2))$ must have a nontrivial kernel so $h^0(\mathcal{I}_C(2)) > 0$. Therefore C lies in some surface of degree 2. Can the surface be twice a hyperplane? No. Can the surface be the union of two planes? No. Could the surface by the singular quadric cone Q_{one}? Yes. Could the surface be the nonsingular quadric surface Q_{ns}? Yes.

If C lies on Q_{ns} then it must have a type (a,b) which must satisfy $a+b=6$ and $(a-1)(b-1)=4$. The only solution is $a=b=3$.

The other possibility is that C lies on Q_{one}. One way to understand C is to figure out all divisors on Q. Another way is to compute an exact sequence like the one above. We obtain

$$0 \to H^0(\mathcal{I}_C(3)) \to H^0(\mathcal{O}_{\mathbf{P}^3}(3)) \to H^0(\mathcal{O}_C(3)) \to \cdots$$

As before one sees that $h^0(\mathcal{O}_C(3)) = 15$ and $h^0(\mathcal{O}_{\mathbf{P}^3}(3)) = 20$. Thus $h^0(\mathcal{I}_C(3)) \geq 5$. Let $q \in H^0(\mathcal{I}_C(2))$ be the defining equation of Q_{one}. Then $xq, yq, zq, wq \in H^0(\mathcal{I}_C(3))$. But $h^0(\mathcal{I}_C(3)) \geq 5$ so there exists an $f \in H^0(\mathcal{I}_C(3))$ so that the global sections xq, yq, zq, wq, f are independent. Thus there is an f not in (q). Since $f \notin (q)$ we see that $F_3 = Z(f) \not\supset Q$ so $C' = F_3 \cap Q$ is a degree 6 not necessarily nonsingular or irreducible curve. Since $C \subset F_3$ and $C \subset Q$ it follows that $C \subset C'$. Since $\deg C = 6 = \deg C'$ it follows by an easy exercise that $C = C'$.

Lemma 41.4. *Suppose $C \subset C'$ are both closed subschemes of $\mathbf{P}^n$ with the same Hilbert polynomial. Then $C = C'$.*

Thus in the case that C lies on Q_{one} we see that C is also a complete intersection $C = Q_{\text{one}} \cap F_3$.

Next we comment on the g^1_3 question in this situation. Projection from the cone point to the conic (the base of the cone) induces a g^1_3 on C. So in genus $g=4$ there is a g^1_3. There is one in the case that C lies on Q_{one} (it is not clear that there is *just* one), there are two in the case that C lies on Q_{ns}, and there is a trivial one in the case that C is hyperelliptic.

Proposition 41.5. *Suppose C is a genus 4 nonsingular complete intersection $Q \cap F_3 \subset \mathbf{P}^3$ with Q of degree 2 and F_3 of degree 3. Then $\omega_C \cong \mathcal{O}_C(1)$.*

Recall that if C is of degree d in $\mathbf{P}^2$ then $\omega_C \cong \mathcal{O}_C(d-3)$. This can be seen from an analysis of the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbf{P}^2}|C \to \Omega_C \to 0$$

together with the exact sequence

$$0 \to \Omega_{\mathbf{P}^2}| \to \mathcal{O}_C(-1)^3 \to \mathcal{O}_{\mathbf{P}^2}|C \to 0.$$

We could do the same thing for $C \subset \mathbf{P}^3$.

Proposition 41.6. *Suppose $C = F_e \cdot F_f \subset \mathbf{P}^3$ with F_e and F_f nonsingular of degree e, f, respectively. Then $\omega_C = \mathcal{O}_C(e+f-4)$.*

This was (II, Ex. 8.4 e) and it can be found in my homework solutions.

42 Curves of Genus 5

There are hyperelliptic curves of genus 5. For example a curve of type $(2,6)$ on the quadric surface $Q \subset \mathbf{P}^3$. Are there any more curves of genus 5?

If C is a curve of genus 5 which is not hyperelliptic then ω_C is very ample so there is a canonical embedding $C \hookrightarrow \mathbf{P}^4$ in which C has degree 8. How many quadric hypersurfaces does such a C lie on?

42.1 The space of quadrics containing an embedded genus 5 curve.

There is an exact sequence

$$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_{\mathbf{P}^4}(2)) \to H^0(\mathcal{O}_C(2)) \to \cdots$$

Riemann-Roch implies $h^0(\mathcal{O}_C(2)) = \deg(\mathcal{O}_C(2)) + 1 - g = 16 + 1 - 5 = 12$. Here $\mathcal{O}_C(2)$ has degree 16 since the degree is additive on the class group and C has degree 8 so the divisor $\mathcal{O}_C(1)$ has degree 8. Also $\mathcal{O}_C(2)$ is superspecial since the canonical divisor has degree $2g-2=8$. Combining this with the fact that $h^0(\mathcal{O}_{\mathbf{P}^4}(2)) = 15$ implies $h^0(\mathcal{I}_C(2)) \geq 3$. Thus C lies on a 3 dimensional space of quadric hypersurfaces. Let F_2, F_2', F_2'' be linearly independent quadric hypersurfaces containing C. In the genus 4 case $F_2 \cap F_3$ was a curve C' which we were able to show was equal to C. But in our situation it might be possible that

$$F_2 \cap F_2' = H \cong \mathbf{P}^3.$$

But in that case $C \subset F_2 \cap F_2'$ would be contained in $\mathbf{P}^3$. But this is not true since C is not contained in any linear subspace since the embedding comes from the complete linear system $|\omega_C|$. (If C were contained in a linear subspace then there would be a dependence relation between the linearly independent global sections of ω_C giving rise to the embedding.)

If $F_2 \cap F_2'$ is a hypersurface then it is defined by a single polynomial equation $f=0$. Then f divides the quadratic defining F_2 and the quadratic defining F_2' so f must be linear. But this case was ruled out above. Thus $S = F_2 \cap F_2'$ is a surface. We cannot conclude that $F_2 \cap F_2' \cap F_2''$ is the curve C. It could happen that $S \subset F_2''$ and so $S = F_2 \cap F_2' \cap F_2''$. It could also happen that $F_2 \cap F_2' \cap F_2''$ is just a part S_1 of the surface S. But this gives us a hint as to how to construct curves of genus 5.

42.2 Genus 5 curves with very ample canonical divisor

In $\mathbf{P}^4$ take F_2, F_2', F_2'' three quadric hypersurfaces which are sufficiently general so that $C = F_2 \cap F_2' \cap F_2''$ is a nonsingular curve. (That this can be done appeals to Bertini's theorem.) Given such a curve C then $\deg C = 2^3 = 8$ and $\omega_C = \mathcal{O}_C(2+2+2-4-1) = \mathcal{O}_C(1)$. Therefore C is the canonical embedding of the abstract curve hiding in the shadows behind C. The genus of C is 5 since $2g-2 = \deg C = 8$. This construction gives examples of curves of genus 5 for which ω_C is very ample, i.e., curves of genus 5 which are not hyperelliptic.

Does this type of curve have a g_3^1, i.e., a complete linear system of degree 3 and dimension 1?

Suppose C is an abstract curve of genus 5 which is not hyperelliptic. Embed C in $\mathbf{P}^4$ by its canonical embedding $C \hookrightarrow \mathbf{P}^4$. There are two possibilities:

1. $C = F_2 \cap F_2' \cap F_2''$
2. $C \subset S = F_2 \cap F_2' \cap F_2''$

As seen above case 1 can occur. In case 2, S is a surface of degree 8 [[I missed the proof.]] We do not yet know if case 2 can occur.

Is C trigonal, that is, does C have a g_3^1? To study this question we introduce a new technique. Suppose $C \hookrightarrow \mathbf{P}^4$ is the canonical embedding of C into $\mathbf{P}^4$ as a degree 8 curve. Then there is a g_3^1 iff there exists three points $P, Q, R \in C$ such that $\dim |P+Q+R| = 1$. Given three points P, Q, R Riemann-Roch implies

$$\dim |P+Q+R| = 3+1-5+\dim |K-P-Q-R| = -1+\dim |K-P-Q-R|$$

so $\dim |P+Q+R| = 1$ iff $\dim |K-P-Q-R| = 2$. The condition that $\dim |K-P-Q-R| = 2$ is that there is a 2 dimensional linear system of effective canonical divisors K which contain P, Q, R.

The technique is to translate the condition $\dim |K-P-Q-R| = 2$ into a geometric criterion involving the embedding $C \hookrightarrow \mathbf{P}^4$. Since the embedding $C \hookrightarrow \mathbf{P}^4$ is canonical every effective divisor in the canonical divisor class is the intersection of C with a hyperplane in $\mathbf{P}^4$. We obtain *every* effective divisor because $|K|$ has dimension 4 and the dimension of the space of lines in $\mathbf{P}^4$ is 4. Thus an effective canonical divisor contains P, Q, R iff there is a hyperplane in $\mathbf{P}^4$ containing P, Q, R. Hence there is a 2 dimensional linear system in $\mathbf{P}^4$ containing P, Q, R iff P, Q, R are collinear in $\mathbf{P}^4$. We have thus interpreted $\dim |P+Q+R|$ in terms of the geometry of where P, Q, R lie on C in the canonical embedding. The upshot of this is

Proposition 42.1. *Suppose C is a not a hyperelliptic curve. Then C has a g_3^1 iff there are 3 points $P, Q, R \in C$ which are collinear in the canonical embedding.*

Notice that the proposition is even true without the assumption that C has genus 5.

We return to our situation. Suppose $C \hookrightarrow \mathbf{P}^4$ is the degree 8 canonical embedding of some nonhyperelliptic curve C of genus 5. Suppose $P, Q, R \in C$ are collinear so they all lie on some line L.

First suppose C is the complete intersection $C = F_2 \cap F_2' \cap F_2''$. Then $P, Q, R \in F_2$ and $P, Q, R \in L$ so $L \cap F_2$ contains at least 3 points. If L is not contained in F_2 then $L \cap F_2$ has $(\deg L) \cdot (\deg F_2) = 2$ points so it must be the case that $L \subset F_2$. By similar reasoning we conclude that $L \subset F_2'$ and $L \subset F_2''$ so $L \subset C$, a contradiction. So C does not have a g_3^1.

Next suppose $C \subset S = F_2 \cap F_2' \cap F_2''$. Then it is possible that $L \subset S$ since this leads to no contradiction. So *maybe* there could be a g_3^1 on C, we still do not know.

Next we show that there do exist genus 5 trigonal curves. We construct directly a genus 5 curve C with nontrivial g_3^1. The things to do is look in $\mathbf{P}^2$ for a curve D of degree 5 which has one node and no other singularities.

As an aside we consider a more general problem. Consider singular irreducible curves C of degree d in $\mathbf{P}^2$ containing r nodes and k cusps. A *node* looks locally like $xy = 0$ and a cusp looks like $y^2 = x^3$. What are the possible triples (d, r, k) which can occur? The answer for small d is

d	(d, r, k)
1	(1,0,0)
2	(2,0,0)
3	(3,1,0), (3,0,1)
4	(4,3,0), ...

In general this is an open problem. Solving it is a guaranteed thesis, seven years of good luck, and an... academic position! The complete answer is probably known up to degree 10.

One constraint comes from the fact that if $\tilde{C}$ is the normalization of C then

$$g(\tilde{C}) = \frac{1}{2}(d-1)(d-2) - r - k \geq 0$$

so $r + k \leq \frac{1}{2}(d-1)(d-2)$.

Finding a D as above means finding a $(5,1,0)$. Suppose $\operatorname{char} k \neq 3, 5$. Let

$$f = xyz^3 + x^5 + y^5.$$

The point $x = y = 0$ is a nodal singularity. There are no other singularities. See this by computing $f_x = yz^3 + 5x^4$, $f_y = xz^3 + 5y^4$, and $f_z = 3xyz^2$. For these to all vanish it must be the case that x, y, or z is 0. If x or y is 0 then both x and y are 0 so we recover the nodal singularity. If x and y are nonzero then $z = 0$ so from $f_x = 0$ and $f_y = 0$ it follows that $x = y = 0$, but $x = y = z = 0$ is not a point.

Let $C = \tilde{D}$ be the normalization of C. Then C is a genus 5 nonsingular curve. (That the normalization of a curve is nonsingular is a fact from commutative algebra.) Pick a node P on D. Then lines through P in $\mathbf{P}^2$ give a map $D - \{P\} \to \mathbf{P}^1$. A point Q in $\mathbf{P}^1$ corresponds to a line L through P. Since P is a double point and D has degree 5, we see that L intersects D in 3 other points. These three points map to $Q \in \mathbf{P}^1$. This map extends to a map on C which corresponds to a g^1_3.

We can now say something about the $3g - 3 = 12$ dimensional space of genus 5 curves. They can be divided into 3 nonempty classes: trigonal, hyperelliptic and general ones with no g^1_3. The hyperelliptic curves form a $2g - 1 = 9$ dimensional family. [[The trigonal curves might form an 11 dimensional family??]] It might be the case that every hyperelliptic curve is trigonal.

Next time we will do the case of genus 6 since a new phenomenon appears.

43 Homework Assignment

Do the following from the book: IV Ex. 3.6, 3.12, 5.4.

Do any one of the following problems (and try to do one that someone else is not doing!)

1. If C is a hyperelliptic curve of genus $g \geq 2$, find the least possible degree of a very ample divisor on C. Is it independent of the particular hyperelliptic curve chosen?

2. Can every hyperelliptic curve of genus $g \geq 2$ be embedded in $\mathbf{P}^3$, so as to be a curve of bidegree $(2, g+1)$ on a non-singular quadric surface Q?

3. If C is hyperelliptic of genus $g \geq 3$, then C does not have a g^1_3 (without base points).

4. If C is a non-hyperelliptic curve of genus $g \geq 4$ show that C has at most a finite number of g^1_3's.

5. If C is a non-hyperelliptic curve of genus $g \geq 3$, then it admits a very ample divisor of degree $d \leq g + 2$.

44 Curves of genus 6

Examples of curves of genus 6.

(a) degree 5 curve in $\mathbf{P}^2$,

(b) a type $(2,7)$ curve (of degree 9) on the quadric surface Q in $\mathbf{P}^3$,

(c) a type $(3,4)$ curve (of degree 7) on the quadric surface Q in $\mathbf{P}^3$.

Claim 44.1. *The abstract curves C which can be realized in types (a), (b), and (c) are mutually disjoint.*

Since (b) is hyperelliptic it has a g_2^1. Since (c) is trigonal (i.e. it has a g_3^1) by exercise 3 it does not have a g_2^1. Thus (b) and (c) are disjoint classes.

I will next show that (a) has no overlap with (b) and (c). First note that (a) has infinitely many g_4^1's, one for each point. This is because projection through a point gives a degree 4 map to $\mathbf{P}^1$. I will prove that (a) has no g_2^1 or g_3^1.

Lemma 44.2. *If C is a nonsingular curve of* $\deg 5$ *in $\mathbf{P}^2$ then C does not have a g_2^1 or a g_3^1.*

The lemma is a special case of a result of Max Noether:

> "On a plane curve, the only linear systems g_d^r of maximal dimension (so r is maximal with respect to d) are the *obvious* ones."

The obvious linear systems are the ones arising in a natural way by intersecting the curve with a straight line, or with a conic section, or a conic section but fixing one point, and so on. Suppose C has degree 5 in $\mathbf{P}^2$. Then cutting with a line gives a g_5^2 since the lines in $\mathbf{P}^2$ form a 2 parameter family and they intersect C in 5 points. Cutting with a conic gives a g_{10}^5 since the conics form a 5 parameter family. By fixing two points one obtains a g_9^4, by fixing three points a g_8^3 and similarly a g_7^2 and g_6^1. All lines with 1 fixed point gives a g_6^2. More generally, if C has degree n in $\mathbf{P}^2$ then there exists a g_{n-1}^1 but Noether's result implies that there does not exist a g_d^1 for $d < n-1$.

Proof. Now we prove the lemma. Let C be a nonsingular curve of degree 5 in $\mathbf{P}^2$. Then $\omega_C = \mathcal{O}_C(d-3) = \mathcal{O}_C(2)$. This curve is *subcanonical*, i.e., $\omega_C = \mathcal{O}_C(\ell)$ for some $\ell > 0$. Thus the canonical embedding is obtained by following $C \hookrightarrow \mathbf{P}^2$ by the 2-uple embedding.

From last time we know that there exists a g_3^1 on C iff there exists points $P, Q, R \in C$ such that P, Q, R are collinear in the canonical embedding. This might lead to a proof but I can not think of it right now, so forget it!

C is hyperelliptic iff there exists P, Q such that $\dim|P+Q| = 1$. By Riemann-Roch

$$\dim|P+Q| = 2 + 1 - 6 + \dim|K - P - Q|$$

so in this situation $\dim|K-P-Q| = 4$. Since C has degree 5 and the canonical divisor has degree 10, the canonical divisor is cut by conics. To see this note that $|K| = g - 1 = 5$ and the dimension of the space of conics in $\mathbf{P}^2$ is also 5. If $\dim|K-P-Q| = 4$ then there exists $P, Q \in C$ such that the family

$$\{ \text{ conics containing } P, Q \ \}$$

has dimension 4. But the family of all conics in $\mathbf{P}^2$ has dimension 5, the family of conics through one fixed point has dimension 4, and the family of conics through two fixed points has dimension 3. Thus C can not be hyperelliptic.

C is trigonal iff there exists points P, Q, R such that $\dim|P+Q+R| = 1$. By Riemann-Roch this latter condition implies that $\dim|K-P-Q-R| = 3$. This would mean that we could find three points P, Q, R in C such that the dimension of the family of conics containing P, Q, R is 3. But the family of conics through P and Q has dimension 3 and there are conics passing through P and Q but not through R so the family of conics through all of P, Q, R must have dimension less than 3. Thus C is not trigonal and the lemma is proved. □

Can every curve of genus 6 be realized as one of type (a), (b), or (c)? The answer is no.

g	g^r_d's
$g=2$	$\exists g^1_2$
$g=3$	$\exists g^1_3$
$g=4$	$\exists$ finitely many g^1_3
$g=5$	$\exists$ infinitely many g^1_4 (not shown in class)
$g=6$	we should *expect* that the general curve has only finitely many g^1_4's

Kleiman-Lacksov prove that what we expect is actually the case in general. Our examples (a), (b), and (c) all have infinitely many g^1_4's so we suspect that our examples do not cover all genus 6 curves.

There exists a curve of a fourth type (d)=none of the above, and this will be the general curve. It is hard to get our hands on a general curve since any time we explicitly make a curve it has special properties and is thus not general.

We want to find a plane curve of degree 6 with 4 nodes and no other singularities. The space of all curves of degree 6 has dimension $\frac{1}{2}d(d+3)=27$. It takes 3 linear conditions to force a node at a particular point, thus it takes 12 linear conditions to get 4 nodes. Since $12<27$ there exist such curves. The curve could have weird singularities, but there is a way to get the curve we want. This is like homework IV.5.4. Thus suppose $C_0 \subset \mathbf{P}^2$ is a curve of degree 6 with exactly 4 singularities which are all nodes. Let $C=\tilde{C_0}$ be the normalization of C_0. Then

$$g=g(C)=\frac{1}{2}(d-1)(d-2)-4=10-4=6.$$

There are five obvious g^1_4's. Four come from projecting away from any of the double points. To get the fifth consider conics passing through all four double points. Thus C has g^1_4's but none of our previous curves did and so the class (d) is nonempty.

Another question is the following.

Question 44.3. *When g is even there are only finitely many g^1_{g-1}'s. For example when $g=4$ there are two g^1_3's, and when $g=6$ there are five g^1_4's. In general how many g^1_{g-1}'s are there?*

We could have also approached showing classes (a), (b), (c), and (d) are disjoint by asking: what is the least degree of a very ample divisor? For (a) the least degree is 5, for (b) it is 9, for (c) it is 7, and for (d) it is 8. We give no proof of this here. This type of classification method should generalize to arbitrary genus.

45 Oral Report Topics

During the week of the 29th of April oral reports will be presented by the students. The suggested topics are

- Curves $/k$ where k is not necessarily algebraically closed and rational points on curves over finite fields.
- The variety of moduli $\mathcal{M}_g$.
- Duality for a finite smooth morphism $X \to Y$.

- Jacobian variety of a curve. ("what does it really mean?")
- Curves on a nonsingular cubic surface in $\mathbf{P}^3$ (ch 5, sec 4).
- Flat families of curves in $\mathbf{P}^3$ (ch 3, sec 9).

Your oral reports must be 20 minutes in length. They should contain precise definitions, statements of the main theorems, some examples, and maybe some proofs if there is time. You should consult with me before you begin.

For the rest of the semester we will be studying curves of genus g, g general and elliptic curves.

46 Curves of general genus

Today's lecture contains hints for some of the homework problems.

Proposition 46.1. *If C is hyperelliptic of genus $g \geq 2$, then the g_2^1 is unique. Furthermore, $K \sim (g-1)D$ for any $D \in g_2^1$.*

Proof. The complete linear system $|K|$ has no base points iff $\dim|K-P| = \dim|K|-1$ for any point P on C. By Riemann-Roch $\dim|K| = 2g-2+1-g = g-1$ and $\dim|P|-\dim|K-P| = 2-g$ so $\dim|K-P| = \dim|P|+g-2$. Thus $\dim|K-P| = \dim|K|-1$ for all P iff $\dim|P| = 0$ for all P which is true since $g \geq 1$. (If there were a one dimensional space of points linearly equivalent to a given point P then C would have genus 0.) Thus K defines a morphism (which is not necessarily an embedding)

$$C \to \mathbf{P}^{g-1}.$$

The map is into $\mathbf{P}^{g-1}$ since $\dim|K| = g-1$. We saw before that if C is not hyperelliptic then this is an embedding. If C is hyperelliptic then the canonical divisor can not be very ample so this map will not be an embedding.

First consider the special case $g=1$. Then we have a map $C \to \mathbf{P}^0 = \{\text{ point }\}$ which is not interesting.

Next suppose $g=2$. Then the canonical divisor induces a map

$$C \to \mathbf{P}^1$$

and K is the pullback of $\mathcal{O}(1)$. Since $\deg K = 2g-2 = 2$ the map $C \to \mathbf{P}^1$ has degree 2, and the complete linear system $|K|$ is a g_2^1. In fact, any $|D|$ is a g_2^1 then $D \sim K$. To see this suppose P, Q are two points and $\dim|P+Q| = 1$. By Riemann-Roch,

$$1 = \dim|P+Q| = 2+1-2+\dim|K-P-Q|$$

so $\dim|K-P-Q| = 0$. Thus there is an effective divisor linearly equivalent to the degree 0 divisor $K-P-Q$. But the only effective divisor of degree 0 is the 0 divisor. Thus $K \sim P+Q$. Thus in genus 2 *any* g_2^1 is given by K since any effective D giving a g_2^1 is linearly equivalent to K.

Suppose now that $g \geq 3$. The complete linear system $|K|$ gives a morphism

$$\varphi : C \to \mathbf{P}^{g-1}.$$

Since K is not very ample, φ is not a closed immersion. Now assume C is hyperelliptic. Suppose P and Q are two points on C such that $P+Q$ lies in some g^1_2. Then by Riemann-Roch,

$$1 = \dim|P+Q| = 2+1-g+\dim|K-P-Q|.$$

Thus $\dim|K-P-Q| = g-2$. But $\dim|K| = g-1$ and $|K|$ has no base points so $\dim|K-P| = g-2$. Thus $\dim|K-P| = \dim|K-P-Q|$ so Q is a base point of $|K-P|$. This means that any canonical divisor containing P also contains Q. Therefore $\varphi(P) = \varphi(Q)$ if we look at this in terms of the morphism. Thus if $P+Q$ is in some g^1_2 then $\varphi(P) = \varphi(Q)$.

Thus φ is at least two-to-one. Let $C_0 = \varphi(C) \subset \mathbf{P}^{g-1}$ be the image curve. Pulling back $\mathcal{O}(1)$ on $\mathbf{P}^{g-1}$ gives $\mathcal{O}_{C_0}(1)$ on C_0 which pulls back to the canonical divisor $\mathcal{O}_C(1) = K$ which has degree $2g-2$. There are two numbers to consider. First the degree of the curve C_0, call it d. Let e be the degree of the finite morphism φ. Then

$$2g-2 = \deg|K| = de.$$

See this by noting that $\mathcal{O}_{C_0}(1)$ is obtained by cutting C_0 with a hyperplane and seeing that it intersects in d points then pulling these points back to their de preimages to obtain the de degree divisor $\mathcal{O}_C = K$.

Since $e \geq 2$ the equality $de = 2(g-1)$ implies $d \leq g-1$. Consider C_0 which is a (possibly singular) integral curve. Let $D_0 = \mathcal{O}_{C_0}(1)$, then D_0 has degree d and since $|D_0|$ gives the embedding of C_0 into $\mathbf{P}^{g-1}$ and C_0 lies in no linear subspace, $\dim|D_0| \geq g-1$. By a previous proposition (IV, Ex. 1.5), we know that $\dim|D_0| \leq \deg D_0$ with equality iff $C_0 \cong \mathbf{P}^1$. But

$$g-1 \leq \dim|D_0| \leq \deg D_0 = d \leq g-1$$

so we do have equality and $C_0 \cong \mathbf{P}^1$ is nonsingular and is in fact the $d-1$-uple embedding of $\mathbf{P}^1$ into $\mathbf{P}^{g-1}$. Furthermore, since $d = g-1$ we also see that $e = 2$.

The upshot of all this is that if C is hyperelliptic and $g \geq 3$ then $|K|$ gives an embedding $\varphi : C \to \mathbf{P}^{g-1}$ which factors through the $g-1$-uple embedding of $\mathbf{P}^1$. Thus $\varphi : C \to \mathbf{P}^{g-1}$ can be written as a composition

$$C \xrightarrow{g^1_2} C_0 = \mathbf{P}^1 \hookrightarrow \mathbf{P}^{g-1}.$$

[[One shows that the g^1_2 is uniquely determined, etc. as on page 343 of the book. WRITE THIS UP HERE!]] $\square$

I thought of another proof of the uniqueness of g^1_2's. It is not in the book. This technique is useful for the exercises. The drawback is that it does not explicitly give K.

Theorem 46.2. *If C is a curve of genus $g \geq 2$ then C can not have two distinct g^1_2's.*

Proof. The proof is by contradiction. Suppose C has non-linearly equivalent divisors D and D' such that $|D|$ and $|D'|$ are g^1_2's. Then $|D|$ and $|D'|$ give morphisms p and p' to $\mathbf{P}^1$ Taking the product gives a morphism $\varphi = p \times p' : C \to \mathbf{P}^1 \times \mathbf{P}^1$. Since $|D|$ and $|D'|$ do not contain any of the same divisors the morphisms p and p' collapse different points and so they are different. Let

$$C_0 = \varphi(C) \subset \mathbf{P}^1 \times \mathbf{P}^1 = Q \subset \mathbf{P}^3$$

then C_0 is a possibly singular but still integral curve on the quadric surface Q. By construction each ruling gives a g^1_2 on C.

Let (a,b) be the bidegree of C_0 on Q. It is clear that C_0 is not a point since p and p' are not constant. Let e be the degree of $\varphi : C \to C_0$. A divisor of type $(1,0)$ on Q pulls back to a divisor on C_0 of degree a. Since the morphism $C \to C_0$ has degree e this divisor of degree a pulls back to a degree ea divisor on C. On the other hand a divisor of type $(1,0)$ on Q pulls back to a divisor of degree 2 since it gives rise to p. Thus $2 = ae$ and similarly $2 = be$.

Case 1: First suppose $e = 2$ and hence $a = b = 1$. Then C_0 is of type $(1,1)$ so the normalization of C_0 is of genus 0. But the arithmetic genus of C_0 is $(1-1)(1-1) = 0$ so, since the genus can only go now upon normalization, we see that C_0 must already be nonsingular. Thus φ maps C into $\mathbf{P}^1$. But p is just the composition of φ with the first projection $Q \to \mathbf{P}^1$ and this projection is the identity $C_0 = \mathbf{P}^1 \subset Q \to \mathbf{P}^1$. Thus p and p' collapse the same points, a contradiction.

Case 2: Next suppose $e = 1$ and hence $a = b = 2$. Then $\varphi : C \to C_0$ induces a birational morphism to the normalization of C_0. But the normalization of C_0 has genus less than or equal to $p_a(C_0) = (2-1)(2-1) = 1$ so the genus of C is less than or equal to 1. In fact, if the genus of C is one then there are two distinct g^1_2's. But under our assumption that $g \geq 2$ we have the desired contradiction. $\square$

The above proof probably can be generalized to show that, for large enough genus, C can not have any g^1_d's.

As in case 1 we saw that a type (1,1) curve C_0 can not be singular since the normalization would then have negative genus. This might help with problems 3 and 4.

Next we give a statement of the theorem which will be proved next time.

Theorem 46.3 (Halphen). *Let C be a curve of genus $g \geq 2$. Then there exists a very ample nonspecial divisor D of degree d iff $d \geq g + 3$.*

The existence is actually very strong in the sense that a Zariski open subset has the property.

This generalizes our proof that $g = 3$ implies that there exists a very ample divisor of degree 6. Note that $2g + 1 = 7$.

There can be very ample *special* divisors of degree less than $g + 3$.

47 Halphen's Theorem

Today we will give one more general result about curves.

Theorem 47.1 (Halphen). *Suppose C is a curve of genus $g \geq 2$. Then there exists a very ample nonspecial divisor D of degree d iff $d \geq g + 3$.*

Proof. ($\Rightarrow$) Suppose D is a very ample nonspecial divisor of degree d. We will show that $d \geq g + 3$. By Riemann-Roch $\ell(D) \geq d + 1 - g$. Since D is very ample D gives rise to an embedding $C \hookrightarrow \mathbf{P}^n$. If $n = 1$ then C has genus 0, a contradiction. If $n = 2$ then somehow [I can not figure this out from my notes], this implies $g < 2$, a contradiction. Thus D gives an embedding $C \hookrightarrow \mathbf{P}^n$ with $n \geq 3$. Thus $\ell(D) \geq 4$ and so $d + 1 - g \geq 4$ so $d \geq g + 3$ as claimed.

($\Leftarrow$) This direction is a little harder. Assume $d \geq g + 3$ and look for a very ample nonspecial divisor D. Remember our criterion for when a divisor is very ample? A divisor D is very ample iff $\dim |D - P - Q| = \dim |D| - 2$ for all points P, Q. By Riemann-Roch this is equivalent to the assertion that for all points P, Q,

$$\dim |K - D| = \dim |K - D + P + Q|.$$

As a "warm-up exercise" we fix d and ask the following. Does there exist a nonspecial effective divisor D of degree d?

Recall:

- For D to be *nonspecial* means that $\dim |K - D| = -1$.
- For D to be *special* means that $K - D$ is linearly equivalent to an effective divisor. Thus D is special iff there exists an effective divisor E such that $D + E \sim K$, i.e., $D + E$ is a canonical divisor.

The last condition shows that D is special iff D is "inside" *some* canonical divisor.

Since $\dim |K| = g - 1$ there is a $g - 1$ dimensional family of effective canonical divisors. From each one there are only finitely many ways to get a special divisor. Thus the dimension of the family of all special divisors of degree d is at most $g - 1$. This shows that if $d \geq g$ then there exists nonspecial divisors of degree d. In fact, most are nonspecial.

Next we prove something which is more difficult. We want to show that there exists a nonspecial very ample divisor of any given degree $d \geq g + 3$. It is enough to prove that the collection of nonspecial not very ample divisors has dimension $\leq g + 2$.

Suppose D is nonspecial. Then for D to be not very ample means that there exists points P and Q so that

$$\dim |D - P - Q| > \dim |D| - 2.$$

A straightforward check using Riemann-Roch shows that this is equivalent to

$$\dim |K - D| < \dim |K - D + P + Q|.$$

Since D is assumed nonspecial $\dim |K - D| = -1$ thus

$$\dim |K - D + P + Q| \geq 0.$$

So there exists an effective divisor E which is special and has degree $d - 2$ such that $E \sim D - P - Q$. [[I thought I saw this yesterday but not even this makes any sense today.]]

By the above work the dimension of the set of effective special divisors is $\leq g - 1$. Thus the set

$$\{\ E + P + Q : E \text{ is effective, special, degree } d - 2,\ P, Q \text{ any points }\}$$

has dimension $\leq g + 1$. But there is another wrinkle. We must count all divisors linearly equivalent to any such $E + P + Q$.

Somehow [[and I haven't figured out how!!]] Hartshorne counts this and concludes that it has dimension $\leq g + 2$. [[I thought sort of hard about this and can not see it, but I got confused at this point in class when I was taking notes so that may be why my notes do not reveal the truth.]] □

48 Hurwitz's Theorem

Suppose X and Y are curves (nonsingular, projective, over an algebraically closed field k). Suppose $f : X \to Y$ is a finite morphism. Then f induces a map of function fields $K(Y) \hookrightarrow K(X)$ which makes $K(X)$ into a finite extension of $K(Y)$. The *degree* of f is the degree of the corresponding extension of function fields $K(Y)/K(X)$. We say f is *separable* (not to be confused with separated) if $K(X)$ is a separable field extension of $K(Y)$.

Suppose $P \in X$ maps to $Q \in Y$. Then there is an induced map of local rings $f^{\#} : \mathcal{O}_Q \hookrightarrow \mathcal{O}_P$. Since X and Y are nonsingular curves this is an extension of discrete valuation rings. Let $t \in \mathcal{O}_Q$ be a uniformizing parameter for $\mathcal{O}_Q$ and let $u \in \mathcal{O}_P$ be a uniformizing parameter for $\mathcal{O}_P$. Then $f^{\#}(t) \in \mathbf{m}_P \subset \mathcal{O}_P$. The *ramification index* of P over Q is $e_P := v_P(f^{\#}(t)) \geq 1$.

We say that a point P lying over a point Q is *wildly ramified* if $\operatorname{char} k = p > 0$ and $p|e$. Otherwise the ramification is called *tame*, i.e., when $\operatorname{char} k = 0$ or $\operatorname{char} k = p \not| e$. Note that we do not, as in some definitions, need to worry about the extension of residue fields being separable because k is algebraically closed.

Theorem 48.1 (Hurwitz). *Suppose $f : X \to Y$ is a finite separable morphism of curves which is at most tamely ramified. Then*

$$2g(X) - 2 = (deg f)(2g(Y) - 2) + \sum_{P \in X} (e_P - 1).$$

Note that as a consequence of this theorem, a finite separable tamely ramified morphism of curves has at most finitely many points of ramification. "Any good theorem has a counterexample."

*Example 48.2 **(Frobenius map)**.* If the morphism f is not separable then there can be infinitely many points of ramification. For example, let k be a field of characteristic p. The divisor corresponding to the invertible sheaf $\mathcal{O}_{\mathbf{P}^1}$ gives a map from $\mathbf{P}^1 \to \mathbf{P}^1$. It is given explicitly simply by

$$(x : y) \mapsto (x^p : y^p).$$

[[Is this true, or do I just want it to be true?]] For any point $P \in \mathbf{P}^1$ we have that $e_P = p$ since the map corresponds to the map

$$k(t) \to k(t) : t \mapsto t^p.$$

Thus this map is widely ramified everywhere.

Hurwitz's theorem. [[First part omitted.]]

Thus we have the exact sequence

$$0 \to f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$$

and $\Omega_{X/Y}$ is a torsion sheaf so it equals $\oplus_{P \in X} (\Omega_{X/Y})_P$. Next we study $\Omega_{X/Y}$ locally. Let P be a point in X lying over Q in Y. Then there is an exact sequence

$$0 \to f^*\Omega_Q \to \Omega_P \to \Omega_{P/Q} \to 0.$$

Here $\Omega_{P/Q}$ is the module of differentials of the local ring $\mathcal{O}_P$ over $\mathcal{O}_Q$. Let t be a uniformizing parameter for $\mathcal{O}_Q$ and let u be a uniformizing parameter for $\mathcal{O}_P$. Then Ω_Q is a free $\mathcal{O}_Q$-module of rank 1 locally generated by dt.

Next let $e = e_P = v_P(t)$, thus $t = au^e$ in $\mathcal{O}_P$ (with a a unit in $\mathcal{O}_P$). Differentiate to see that

$$dt = aeu^{e-1}du + u^e da.$$

The term $u^e da$ is some mysterious element of Ω_P. We do not know anything about what a looks like so da can be very strange. If $e = 0$ in k this means that dt would be *wild* in the sense that we have no real control over da. Since we are assuming that the ramification is tame, $\operatorname{char} k \not| e$, so $e \neq 0$ in k. Thus $aeu^{e-1}du \neq 0$. Define $b \in \mathcal{O}_Q$ by $u^e da = bdu$. Since $v_p(b) \geq e$ and $v_p(aeu^{e-1}) = e - 1$ we see that if $dt = Adu$ then $v_p(A) = e - 1$. This means that $u^{e-1}du$ is zero in $\Omega_{P/Q}$ [[but why is no lower power of u times du also 0.]] Thus $\Omega_{P/Q}$ is a principal module generated by du of length $e - 1$. Thus noncanonically

$$\Omega_{P/Q} \cong \mathcal{O}_P/u^{e-1}.$$

We thus have an exact sequence

$$0 \to f^*(\Omega_Y) \to \Omega_X \to R \to 0$$

where $R = \oplus_P \mathcal{O}_P/u_P^{e_P-1}$. Let K_X denote the canonical divisor on X and K_Y be the canonical divisor on Y. Then $\Omega_X = \mathcal{L}(K_X)$ and $\Omega_Y = \mathcal{L}(K_Y)$. □

Example 48.3. Assume $\operatorname{char} k \neq 2$. Let C be the cubic curve in $\mathbf{P}^2$ defined by $y^2 = x(x+1)(x-1)$. Let π by the degree 2 projection of C from ∞ onto $\mathbf{P}^1$ (i.e., the x-axis). There are 4 points of ramification, namely $(-1,0),(0,0),(1,0)$, and ∞. Hurwitz's theorem is satisfied since

$$2 \cdot 1 - 2 = 2(2 \cdot 0 - 2) + \sum_{4 \text{ points}} (2-1).$$

Example 48.4. Let C be a cubic curve in $\mathbf{P}^2$. Let $\mathcal{O}$ be a point not on C and fix a copy of $\mathbf{P}^1 \subset \mathbf{P}^2$. Then projection from $\mathcal{O}$ onto $\mathbf{P}^1$ defines a degree 3 map $C \to \mathbf{P}^1$. This map is ramified at a point $P \in C$ exactly when the line from $\mathcal{O}$ to P is tangent to C. Assume there are no inflectional tangents so that the ramification degree of ramified points is 2. By Hurwitz's theorem,

$$0 = 3(-2) + \sum (e_P - 1)$$

so there are 6 points of ramification 2. This means that from $\mathcal{O}$ one can draw 6 tangent lines to C.

Example 48.5. Let C be a cubic curve in $\mathbf{P}^2$. Define a map $\varphi : C \to \mathbf{P}^1$ by sending a point P to the intersection of the tangent space to C at P with $\mathbf{P}^1 \subset \mathbf{P}^2$. By the above example this map has degree 6. By Hurwitz's theorem,

$$0 = 6(-2) + \sum (e_P - 1)$$

so there are 12 points of ramification. [[Of course one must show (or set things up so) that the ramification of a point is at most 2.]] There are 3 ramification points where $\mathbf{P}^1$ intersects C. The other 9 ramification points are inflection points. Thus Hurwitz's theorem implies that there are 9 inflections on a cubic.

49 Elliptic Curves

Definition 49.1. An elliptic curve C is a nonsingular projective curve of genus 1.

We know that any divisor of degree $d \geq 3$ is very ample. Thus there is an embedding $C \hookrightarrow \mathbf{P}^2$ of C into $\mathbf{P}^2$ as a degree 3 curve.

Theorem 49.2. *Suppose* $\operatorname{char} k \neq 2$ *and k is algebraically closed. If C is an elliptic curve then there is an embedding $C \hookrightarrow \mathbf{P}^2$ with equation*

$$y^2 = x(x-1)(x-\lambda)$$

for some $\lambda \in k$, $\lambda \neq 1, 0$.

Proof. Fix a point $P_0 \in C$ and let $D = 3P_0$. Then, after choosing section $x_0, x_1, x_2 \in H^0(\mathcal{L}(3P_0))$ we obtain an embedding $C \hookrightarrow \mathbf{P}^2$. Construct basis for the global sections of $\mathcal{L}(nP_0)$ for various n.

	$\mathcal{O}_C$	$\mathcal{L}(P_0)$	$\mathcal{L}(2P_0)$	$\mathcal{L}(3P_0)$	$\mathcal{L}(4P_0)$	$\mathcal{L}(5P_0)$	$\mathcal{L}(6P_0)$
h^0	1	1	2	3	4	5	6
basis of H^0	1	1	$1, x$	$1, x, y$	$1, x, y, x^2$	$1, x, y, x^2, xy$	$1, x, y, x^2, xy, x^3, y^2$

Since the seven function $1, x, y, x^2, xy, x^3, y^2$ lie in a six dimensional space there must be a dependence relation

$$ay^2 + bx^3 + cxy + dx^2 + ey + fx + g = 0$$

for some $a, b, c, d, e, f, g \in k$. Furthermore, both x^3 and y^2 occur with coefficient not equal to zero, because they are the only functions with a 6-fold pole at P_0. So replacing y be a suitable scalar multiple we may assume $a = 1$. Preparing to complete the square we rewrite the relation as

$$y^2 + (cxy + ey) + (\frac{1}{2}cx + \frac{1}{2}e)^2 - (\frac{1}{2}cx + \frac{1}{2}e)^2 + bx^3 + dx^2 + fx + g = 0.$$

Replacing y by $\frac{1}{2}cx + \frac{1}{2}e$ transforms the equation into

$$y^2 = e(x-a)(x-b)(x-c)$$

where $a, b, c, e \in k$ are new constants. Next absorb e to obtain

$$y^2 = (x-a)(x-b)(x-c).$$

Now translate x by a to obtain an equation of the form

$$y^2 = x(x-a)(x-b).$$

Multiply and divide by a^3 to obtain

$$y^2 = a^3\frac{x}{a}(\frac{x}{a} - 1)(\frac{x}{a} - \frac{b}{a}).$$

Replace x by $\frac{x}{a}$ and absorb a^3 to get

$$y^2 = x(x-1)(x-\lambda)$$

where $\lambda \neq 0, 1$ since C is nonsingular. $\square$

In this proof we definitely used that k is algebraically closed to absorb constants and that the characteristic is not 2 in order to complete the square.

Remark 49.3. Since $A = k[x,y]/(y^2 - x(x-1)(x-\lambda))$ is not a UFD the class group of C is nontrivial.

How unique is the representation $y^2 = x(x-1)(x-\lambda)$? The answer is that it is not unique. For example, replace x by $x+1$ to obtain

$$y^2 = (x+1)x(x+1-\lambda)$$

then divide by $-(1-\lambda)$ to obtain

$$y^2 = -(\frac{x}{1-\lambda} + \frac{1}{-\lambda}\frac{x}{1-\lambda}(\frac{x}{1-\lambda} + 1).$$

Next replace x by $-\frac{x}{1-\lambda}$ and absorb the minus sign to obtain

$$y^2 = x(x-1)(x - \frac{1}{1-\lambda}).$$

By similar methods we get any of six choices:

$$\lambda \mapsto \lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, 1-\frac{1}{\lambda}.$$

Proposition 49.4. *In the representation C in the form*

$$y^2 = x(x-1)(x-\lambda)$$

, λ up to the group S_3 depends only on C.

The first thing to struggle with is the dependency on the choice of P_0.

50 Automorphisms of Elliptic Curves

Let C be an elliptic curve so C has genus 1. Suppose $\operatorname{char} k \neq 2$. Fix a point P_0 on C. Then the linear system $|3P_0|$ gives an embedding

$$|3P_0| : C \hookrightarrow \mathbf{P}^2.$$

(The divisor $3P_0$ is very ample by (3.3.3) and Riemann-Roch implies $\dim |3P_0| = 2$.) For suitable choice of coordinates C is given by

$$y^2 = x(x-1)(x-\lambda), \quad \lambda \in k, \quad \lambda \neq 0, 1.$$

The choice of λ is not unique since S_3 acts by

$$\lambda \mapsto \lambda, \frac{1}{\lambda}, 1-\lambda, 1-\frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}.$$

Except for this ambiguity, λ is uniquely determined. Define

$$j(\lambda) = \frac{2^8(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}.$$

Then

Theorem 50.1. *The j-invariant has the following properties.*

- *$j \in k$,*
- *j depends only on C,*
- *$C_1 \cong C_2$ iff $j(C_1) = j(C_2)$, and*
- *for all $j \in k$ there is a curve C such that $j(C) = j$.*

The theorem implies that there is a bijection of sets

$$\left\{ \text{ isomorphism classes of elliptic curves } / \ k \ \right\} \xrightarrow{\sim} \left\{ \text{ elements of } k \ \right\}.$$

This can be given a moduli theoretic interpretation.

Example 50.2. Let E be the elliptic curve defined by

$$y^2 = x(x-1)(x+1) = x^3 - x.$$

Thus $\lambda = -1$ so

$$j = \frac{2^8(3)^3}{1 \cdot 2^2} = 2^6 \cdot 3^3 = 12^3 = 1728.$$

If C is an elliptic curve then $\operatorname{Aut} C$ is transitive. So to study the automorphisms of C we need to cut down the number under consideration. To do this fix a point P_0 and consider

$$\operatorname{Aut}(C, P_0) = \{ \text{ automorphisms of } C \text{ fixing } P_0 \ \}.$$

Suppose C is given by the equation

$$y^2 = x(x-1)(x-\lambda)$$

and $P_0 = (0:1:0)$ is the point at infinity. Then the map σ defined by $y \mapsto -y$ and $x \mapsto x$ is an element of $\operatorname{Aut}(C, P_0)$. Note that σ is the covering transformation of some 2-to-1 map $C \to \mathbf{P}^1$.

Is σ unique? Let P be any point on C such that $Q = \sigma(P) \neq P$. Then σ is a covering transformation of the morphism determined by the linear system $|P+Q|$. Associated to any degree 2 map to $\mathbf{P}^1$ there is a covering transformation (the hyperelliptic involution). Thus if $R+S$ is an effective divisor of degree 2 which is not linearly equivalent to $P+Q$ then we obtain a different degree 2 morphism to $\mathbf{P}^1$ and hence a different automorphism of degree two. The conclusion in class was that σ is *not* unique because you obtain many different automorphisms of degree 2 in the above manner. But there is no reason any of these should *fix* P_0. Furthermore, if $\operatorname{Aut}(C, P_0)$ really turns out to be the units in an order in a number field then there σ must be unique because there is only one nontrivial square root of -1.

Let C be as in the above example, so C is defined by

$$y^2 = x(x-1)(x+1).$$

Then the map τ defined by

$$\tau: \quad \begin{cases} x \mapsto -x \\ y \mapsto iy \end{cases}$$

is an automorphism of C fixing $P_0 = (0:1:0)$. Furthermore $\sigma = \tau^2$ so $\operatorname{Aut}(C, P_0)$ contains at least $\mathbf{Z}/4\mathbf{Z}$. Is $\operatorname{Aut}(C, P_0)$ exactly $\mathbf{Z}/4\mathbf{Z}$? We will come back to this question later.

The map sending every point to its double under the group law is not an automorphism although it does fix P_0. Since k is algebraically closed there are 4 points on C which map to P_0 under multiplication by 2.

Example 50.3. Let C be the genus 1 curve in $\mathbf{P}^3$ defined by

$$x^3 + y^3 + z^3 = 0.$$

Suppose $\operatorname{char} k \neq 2, 3$. We want to find an equation for C of the form

$$y^2 = x(x-1)(x-\lambda).$$

Since ∞ is an inflectional tangent of $y^2 = x(x-1)(x-\lambda)$ we hunt for an inflectional tangent of C. The line $y+z=0$ meets C in an inflectional tangent since

$$x^3 + y^3 + z^3 = x^3 + (y+z)(y^2 - yz + z^2)$$

so setting $y+z=0$ gives $x^3 = 0$.

Next perform the change of variables $z = z' - y$. The reason for doing this is so that $z' = 0$ will give the point of intersection of C with $y+z=0$. After substitution the equation becomes

$$x^3 + y^3 + (z' - y)^3 = 0$$

which is

$$x^3 + y^3 + z'^3 - 3z'^2 y + 3z' y^2 - y^3 = 0.$$

Setting $z' = 1$ we obtain

$$x^3 - 3y + 3y^2 + 1 = 0$$

or equivalently, after factoring the -1 into x^3,

$$y^2 - y = x^3 - \frac{1}{3}.$$

Next working modulo linear changes of variables gives

$$y^2 - y + \frac{1}{4} = x^3 - \frac{1}{3} + \frac{1}{4},$$

$$(y - \frac{1}{2})^2 = x^3 - \frac{1}{12},$$

$$y^2 = x^3 - \frac{1}{12},$$

$$y^2 = x^3 - 1,$$

$$y^2 = (x-1)(x-\omega)(x-\omega^2), \quad \text{where } \omega^3 = 1.$$

In general

$$y^2 = (x-a)(x-b)(x-c)$$

is equivalent to $y^2 = x(x-1)(x-\lambda)$ where $\lambda = \frac{a-c}{b-c}$. In our situation this means

$$\lambda = \frac{1-\omega}{\omega^2 - \omega} = \frac{1-\omega}{\omega(\omega - 1)} = -\frac{1}{\omega} = -\omega^2 = \frac{1}{2} + \frac{1}{2}\sqrt{-3}.$$

Thus $j(\lambda) = 0$ and C is defined by

$$y^2 = x(x-1)(x+\omega^2).$$

At this point Hartshorne tried to explicitly describe a degree 3 automorphism of C which fixes infinity. But try as he may it did not come out right. Note that $x \mapsto x$ and $y \mapsto -y$ defines a degree 2 automorphism of C.

To define a degree 3 automorphism look at the Fermat form of C

$$x^3 + y^3 + z^3 = 0, \qquad P_0 = (0:1:-1).$$

Then

$$\sigma : \quad \begin{cases} x \mapsto \omega x \\ y \mapsto y \\ z \mapsto z \end{cases}$$

is an automorphism of degree 3 fixing P_0. Interchanging y and z yields an automorphism of degree 2 fixing P_0 and which commutes with σ. Thus $\mathrm{Aut}(C, P_0)$ is at least $\mathbf{Z}/6\mathbf{Z}$.

Now we approach the automorphism group abstractly.

Theorem 50.4. *If C is an elliptic curve and P_0 is a fixed point then*

$$\mathrm{Aut}(C, P_0) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{if } j \neq 0, 12^3 \\ \mathbf{Z}/4\mathbf{Z} & \text{if } j = 12^3 \neq 0, \ \mathrm{char}\, k \neq 3 \\ \mathbf{Z}/6\mathbf{Z} & \text{if } j = 0 \neq 12^3, \ \mathrm{char}\, k \neq 3 \\ \mathbf{Z}/12\mathbf{Z} & \text{if } j = 0 = 12^3, \ \mathrm{char}\, k = 3 \\ \mathbf{Z}/24\mathbf{Z} & \text{if } j = 0 = 12^3, \ \mathrm{char}\, k = 2 \end{cases}$$

We do not prove the theorem now but the idea is to stare at the following square

$$\begin{array}{ccccc} & C & \xrightarrow{\sigma} & C & \\ |2P_0| & \downarrow & & \downarrow & |2P_0| \\ & \mathbf{P}^1 & \xrightarrow{\exists\, \tau} & \mathbf{P}^1 & \end{array} .$$

Example 50.5. Consider

$$y^2 = x(x-1)(x+1)$$

in characteristic 3. Define automorphisms

$$\tau: \quad \begin{cases} x \mapsto -x \\ y \mapsto iy \end{cases} \qquad \rho: \quad \begin{cases} x \mapsto x+1 \\ y \mapsto y \end{cases}$$

of orders 4 and 3. Clearly $\tau\rho = \rho\tau$ ... but upon checking this, clearly $\tau\rho \neq \rho\tau$. Thus $\mathrm{Aut}(C, P_0)$ is obviously *not* cyclic of order 12, in fact it is an extension of S_3.

Let C be an elliptic curve and P_0 a fixed point on C. Then we associate to the pair C, P_0 the algebraic objects

- $\mathrm{Aut}(C, P_0)$,
- a group structure on C with P_0 as the identity, and
- the endomorphism ring $\mathrm{End}(C, P_0)$.

The ring $\mathrm{End}(C, P_0)$ is defined to be the set of morphisms $\theta : C \to C$ fixing P_0. By general facts about abelian varieties any such θ is in fact a group endomorphism. In characteristic 0 this endomorphism ring is either $\mathbf{Z}$ or an order in an imaginary quadratic number field. In characteristic p the endomorphism ring can be an order in a rank 4 quaternionic extension of $\mathbf{Z}$. In characteristic p the endomorphism ring can not be just $\mathbf{Z}$ because of the Frobenius endomorphism which verifies a certain quadratic equation. [[How does this last statement work?]]

51 Moduli Spaces

For the rest of the semester I am going to talk about Jacobian varieties, variety of moduli and flat families. In each of these situations we parameterize "something or others" by points on a variety then apply algebraic geometry to the variety.

Here are some examples.

Start	Set	Parameter Space
A curve C	$\operatorname{Pic}^0 C = \{$ divisors of degree 0 modulo linear equivalence $\}$ Not just a set but a functor **Sch** $\to$ **Set**	Closed points of Jacobian variety J. J represents the functor.
$\mathbf{P}^n_k$ and $P(z) \in \mathbf{Q}[z]$	set of closed subschemes $Z \subset \mathbf{P}^n_k$ with Hilbert polynomial $P(z)$	The Hilbert scheme $\operatorname{Hilb}^P(\mathbf{P}^n)$
fix $g > 0$	isomorphism classes of curves of genus g	The variety M_g of moduli. Fine if reps. the functor, coarse if not.
$\mathbf{P}^n_k$, degree d, and dimension r	set of cycles of dim. r and degree d in $\mathbf{P}^n$ modulo rational equivalence.	The Chow scheme. It does not represent a functor.

A cycle of dimension r is a $\mathbf{Z}$-linear combination of subschemes of dimension r. If $Z = \sum n_i Y_i$ is a cycle of dimension r then $\deg Z = \sum n_i \deg(Y_i)$.

52 The Jacobian Variety

Fix a curve C of genus g. Then $\operatorname{Pic}^0 C$ is the group of divisors of degree 0 modulo linear equivalence. The Jacobian is going to be a variety J whose closed points correspond to divisors $D \in \operatorname{Pic}^0 C$.

A closed point of J is a map $\varphi : \operatorname{Spec} k \to J$. Taking the product (over $\operatorname{Spec} k$) of this map with the identity map $C \to C$ gives a commuting square

$$\begin{array}{ccc} C & \xrightarrow{\varphi'} & C \times J \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\varphi} & J \end{array}$$

Given an invertible sheaf $\mathcal{D}$ on $C \times J$ and a closed point t of J (given by a morphism $\varphi : \operatorname{Spec} k \to J$) define an invertible sheaf $\mathcal{D}_t$ on C by

$$\mathcal{D}_t = \varphi'^* \mathcal{D}.$$

In this way we obtain a map from the closed points on J to $\operatorname{Pic}(C)$.

We first strengthen our requirement for the Jacobian of C by asking for a divisor $\mathcal{D}$ on $C \times J$ such that the map $\mathcal{D} \mapsto \mathcal{D}_t$ gives a correspondence

$$\text{closed points}(J) \xrightarrow{\sim} \operatorname{Pic}^0(C).$$

Grothendieck's genius was to generalize all of this. The point is to replace $\operatorname{Spec} k$-valued points of J with T-valued points of J where T is any scheme over k. Taking the product of $T \xrightarrow{\varphi} J$ with $C \xrightarrow{\mathrm{id}_C} C$ we obtain a diagram

$$\begin{array}{ccc} C \times T & \xrightarrow{\varphi'} & C \times J \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & J \end{array}$$

If $\mathcal{L}$ is an invertible sheaf on $C \times J$ then associate to $\mathcal{L}$ the invertible sheaf $\mathcal{M} = \varphi'^*\mathcal{L}$.

Next we make a stronger requirement for the Jacobian of C. We require that there exist an invertible sheaf with the following universal property. For any scheme T and any invertible sheaf $\mathcal{M}$ on $C \times T$ which is "of degree 0 along the fibers" there exists a unique morphism $\varphi : T \to J$ such that $\mathcal{M} = \varphi'^*\mathcal{L}$.

The point is that we are parameterizing families of divisor classes. We make the following tentative definition. It will *not* turn out to be the right one.

Definition 52.1. The *Jacobian variety* is a pair $(J, \mathcal{L})$ with $\mathcal{L}$ an invertible sheaf on $C \times J$ such that for all schemes T and for all invertible sheaves $\mathcal{M}$ on $C \times T$ of degree 0 on the fibers, there exists a unique morphism $\varphi : T \to J$ such that $\mathcal{M} \cong \varphi'^*\mathcal{L}$.

What does "degree 0 on the fibers" mean? Suppose $\mathcal{M}$ is an invertible sheaf on $C \times T$. Let t be a point in T. Thus t can be thought of as a map

$$\psi : \operatorname{Spec} \kappa(t) \to T$$

where $\kappa(t)$ is the residue field at t. Tensoring this map with the identity map $\mathrm{id} : C \to C$ yields a map

$$\psi' : C_t = C \times_k \operatorname{Spec} \kappa(t) \to C \times T.$$

If $\mathcal{M}$ is an invertible sheaf, define $\mathcal{M}_t$ to be the pullback $\psi'^*\mathcal{M}$. The degree of $\mathcal{M}_t$ as a divisor on the curve C_t makes sense. Although $\kappa(t)$ may not be algebraically closed we can still define the degree of a divisor in a natural way.

Definition 52.2. $\mathcal{M}$ on $C \times T$ is said to be *of degree* 0 *along the fibers* if for all $t \in T$, $\deg \mathcal{M}_t = 0$.

Remark 52.3. One can show that the map $t \mapsto \deg(\mathcal{M}_t)$ is a continuous function $T \to \mathbf{Z}$.

There is still one subtlety. Our tentative definition of the Jacobian is bad since it obviously cannot exist.

Consider the diagram

$$\begin{array}{ccccc} C \times T & \xrightarrow{\psi'} & C = C_t & \xrightarrow{\varphi'} & C \times J \\ {\scriptstyle p}\downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{\psi} & \operatorname{Spec} k & \xrightarrow{\varphi} & J \end{array}$$

Pullback $\mathcal{L}$ on $C \times J$ via φ' to obtain $\mathcal{L}_t = \varphi'^*\mathcal{L}$. Then pullback $\mathcal{L}_t$ to obtain $\psi'^*\mathcal{L}_t$ on $C \times T$. Let $\mathcal{N}$ be any invertible sheaf on T. Then $\mathcal{F} = p^*\mathcal{N} \otimes \psi'^*\mathcal{L}_t$ is an invertible sheaf on $C \times T$. Furthermore $p^*\mathcal{N}$ is trivial on the fibers so the fibers of $\mathcal{F}$ do not depend on $\mathcal{N}$. [[I can not seem to figure out why this all leads to a contradiction!]] For some reason the map from $T \to J$ which corresponds to $\mathcal{F}$ must be the constant map sending everything to the point corresponding to φ. Since in general there are many possibilities for $p^*\mathcal{N}$ this gives a contradiction. [[No matter what, I can not seem to understand this. I do not see why the map from T to J corresponding to $\mathcal{F}$ must be anything in particular.]]

So to the correct definition is that we require $\mathcal{M}$ to equal $\varphi'^*\mathcal{L}$ in the quotient group

$$\operatorname{Pic}^0(C \times T/T) := \operatorname{Pic}^0(C \times T)/p^* \operatorname{Pic} T.$$

Now we have the correct definition of the Jacobian variety.

This says the pair $(J, \mathcal{L})$ represents the functor

$$(F : \mathbf{Sch}/k)^o \to \mathbf{Set}$$

$$T \mapsto \mathrm{Pic}^0(C \times T/T).$$

Here $(\mathbf{Sch}/k)^o$ is the opposite category of the category of schemes over k obtained by reversing all the arrows. If $\psi : T' \to T$ is a morphism then F sends ψ to the group homomorphism

$$\mathrm{Pic}^0(C \times T/T) \to \mathrm{Pic}^0(C \times T'/T')$$

defined by $\mathcal{M} \mapsto \psi'^*\mathcal{M}$. Here ψ' is the map defined by the diagram

$$\begin{array}{ccc} C \times T' & \xrightarrow{\psi'} & C \times T \\ \downarrow & & \downarrow \\ T' & \xrightarrow{\psi} & T \end{array}$$

To say that $(J, \mathcal{L})$ *represents* F is to say that $\mathrm{Hom}(T, J) = F(T)$ in the sense that the map $\varphi \mapsto \varphi^*\mathcal{L}$ is a bijection.

Existence of the Jacobian variety in general is difficult. When the genus is 1 we are lucky because the Jacobian is just the curve itself. For higher genus we must do something nontrivial. Let me give some consequences of existence.

52.1 Consequence of existence of the Jacobian

1. J is automatically an abelian group scheme.

Definition 52.4. If C is any category with finite products then a group object is $G \in \mathrm{Ob}(C)$ given with morphisms

$$\begin{aligned} \mu : G \times G \to G & \quad \text{(multiplication)} \\ \rho : G \to G & \quad \text{(inverse)} \\ \varepsilon : \{e\} \to G & \quad \text{(identity)} \end{aligned}$$

For any $G \in \mathrm{Ob}\, C$ there exists a natural representable functor $h_G = \mathrm{Hom}(\cdot, G)$. To say that the functor h_G factors through the category $\mathbf{Grp}$ of groups is equivalent to saying that G is a group object. In particular since J represents a functor to the category of abelian groups it is an abelian group scheme.

2. The Zariski tangent space to J at 0.

Let P_0 be the zero point of J. Let $\mathbf{m}_0 \subset \mathcal{O}_{P_0}$ be the maximal ideal of the local ring at P_0. Then the Zariski tangent space at P_0 is the dual vector space $(\mathbf{m}_0/\mathbf{m}_0^2)'$. [[I will figure this out later.]]

53 The Jacobian

In this lecture we reconsider more directly the Jacobian in greater depth. Let C be a curve of genus g. How can we parameterize divisors of degree 0 on C? The following definition gives a notion of a family of divisors of degree 0.

Definition 53.1. If T is any scheme, a *family of divisor classes on C of degree 0* parameterized by T is an element of

$$\operatorname{Pic}^0(C \times T/T) := \operatorname{Pic}^0(C \times T)/p^* \operatorname{Pic}^0 T$$

where $p : C \times T \to T$ is projection.

Definition 53.2. If $\mathcal{M}$ is an invertible sheaf on $C \times T$ then the *fiber $\mathcal{M}_t$ of $\mathcal{M}$ at t* is the pullback of $\mathcal{M}$ by the map $C_t \to C \times T$. The diagram is

$$\begin{tikzcd} C_t \arrow[r] \arrow[d] & C \times T \arrow[d, "p"] \\ \operatorname{Spec} \kappa(t) \arrow[r] & T \end{tikzcd}$$

Definition 53.3. The *Jacobian variety* is a pair $(J, \mathcal{L})$ where J is a scheme over k and $\mathcal{L} \in \operatorname{Pic}^0(C \times J/J)$ such that the pair $(J, \mathcal{L})$ represents the functor

$$T \mapsto \operatorname{Pic}^0(C \times T/T).$$

In other words, for every family $\mathcal{M}$ on $C \times T$ there exists a unique $\varphi : T \to J$ such that $\mathcal{M} = \varphi'^* \mathcal{L}$.

$$\begin{tikzcd} C \times T \arrow[r, "\varphi'"] \arrow[d] & C \times J \arrow[d] \\ T \arrow[r, "\varphi"] & J \end{tikzcd}$$

53.1 Consequences of existence

53.1.1 Group structure

J is automatically an abelian group scheme. In particular,

$$\mu : J \times J \to J$$

is a morphism.

Example 53.4. When $g = 1$ we obtain a group structure on the elliptic curve C.

Let $\mathcal{L}$ be any divisor class of degree 1. By Riemann-Roch $h^0(\mathcal{L}) = 1 + 1 - 1 = 1$ so there is some s which spans $H^0(\mathcal{L})$. This means that there is exactly one effective divisor of degree 1 corresponding to $\mathcal{L}$, namely a point. Thus for any such $\mathcal{L}$ of degree 1 there is a unique point $P \in C$ such that $\mathcal{L} \cong \mathcal{L}(P)$.

Fix a point $P_0 \in C$. Consider the map $C \times C \to C$ defined as follows. Send the pair $\langle P, Q \rangle$ to the point R corresponding to the degree 1 divisor

$$\mathcal{L}(P + Q - P_0) \cong \mathcal{L}(R).$$

The map $\langle P, Q \rangle \mapsto R$ defines a group structure on C. It is *not* obvious that this rule gives a morphism $C \times C \to C$. [[We have only defined a map on closed points. Hmm.]]

53.1.2 Natural fibration, dimension

Fix $d > 0$. Pick a basepoint $P_0 \in C$. Consider the product

$$C^d = C \times \cdots \times C.$$

Its closed points are $(P_1, \ldots, P_d)$ with $P_i \in C$.

Proposition 53.5. *There exists a natural morphism $C^d \to J$. On closed points it is*

$$(P_1, \ldots, P_d) \mapsto \sum_{i=1}^{d} P_i - dP_0.$$

Proof. To give a morphism $C^d \to J$ is equivalent to giving an appropriate family on $C \times T/T$ where $T = C^d$. Let

$$D = \{(R, P_1, \ldots, P_d) : R \text{ is one of the } P_i\}.$$

Since D is the sum of pullbacks of various diagonals it defines a divisor of degree d. Let $p : C \times T \to C$ be projection onto C. Let

$$\mathcal{M} = \mathcal{L}(D) - d(p^*\mathcal{L}(P_0)).$$

[[Probably if someone thinks about it for awhile she sees that $\mathcal{M}$ gives the desired family.]] Check that $\mathcal{M} = \varphi^*\mathcal{L}$ where φ is the map we want on closed points. □

The first interesting case to consider is when $d > 2g - 2$. Then the fiber of a point $j \in J$ of the map

$$\begin{array}{c} C^d \\ \downarrow \\ J \end{array}$$

is

$$C_j^d = \{(P_1, \ldots, P_d) : \sum P_i - dP_0 = j \text{ in } J\ \} = |j + dP_0|.$$

By Riemann-Roch the size of a fiber is thus

$$\dim |j + dP_0| = d + 1 - g - 1 = d - g.$$

Let $C^{(d)}$ by C^d modulo the action of S_d. Then $C^{(d)}$ is of dimension d and the above map factors as a surjection $C^{(d)} \to J$ with fibers of dimension $d - g$. Thus J has dimension g.

A second interesting special case is when $d = g$. If $P_1 + \cdots + P_g$ is chosen sufficiently generally then $|P_1 + \cdots + P_g| = \{P_1 + \cdots + P_g\}$. To see this choose each P_i so that it is not a basepoint of $|K - P_1 - \cdots - P_{i-1}|$. Then $\ell(K - P_1 - \cdots - P_g) = 0$ so by Riemann-Roch

$$\ell(P_1 + \cdots + P_g) = g + 1 - g + \ell(K - P_1 - \cdots - P_g) = 1.$$

This means that the general fiber of $C^{(g)} \to J$ has degree 1. Thus it is a birational map, but it is not necessarily an isomorphism since there can exist special divisors of degree g.

Let U be the open subset of nonspecial divisors on J. This map hints at the construction of the Jacobian since it gives rise to a "germ of the group law" or a "group chunk"

$$U \times U \to U$$

$$\langle D, E \rangle \mapsto D + E - gP_0.$$

The next step is to try to fill out the group law to get a group law on J. This is Weil's method.

53.1.3 The Zariski tangent space

We compute the Zariski tangent space to J at 0.

Definition 53.6. The *Zariski tangent space* to $0 \in J$ is

$$T_{J,0} = (\mathbf{m}_0/\mathbf{m}_0^2)^\vee.$$

Let ε be such that $\varepsilon^2 = 0$. The elements of $T_{J,0}$ are in one-to-one correspondence with morphisms

$$\operatorname{Spec} k[\varepsilon] \to J$$

sending the point of $\operatorname{Spec} k[\varepsilon]$ to $0 \in J$. The corresponding map $\mathcal{O}_0 \to k[\varepsilon]$ sends $\mathbf{m}_0$ to (ε). Since $\varepsilon^2 = 0$ this map factors through $\mathbf{m}_0/\mathbf{m}_0^2$ giving an element of $T_{J,0}$.

Now rewrite $T_{J,0}$ as

$$T_{J,0} = \operatorname{Hom}_0(\operatorname{Spec} k[\varepsilon], J)$$

where Hom_0 means all homomorphisms sending the point of $\operatorname{Spec} k[\varepsilon]$ to $0 \in J$.

The next step is to use the fact that J represents a certain functor. The point is that

$$\operatorname{Hom}_0(\operatorname{Spec} k[\varepsilon], J) = \ker(\operatorname{Pic}^0(C \times T) \to \operatorname{Pic}^0(C)).$$

[[This is a direct check using the fact that $0 \in J$, by representability, must correspond to $\mathcal{O}_C$ on C.]]

The last step is to write down some exact sequences.

54 Flatness

[[The orals are in 959 1-3pm on Monday and Wednesday. Monday Janos, William, then Nghi talk. Wednesday Wayne, Amod, then Matt talk.]]

The outline for this lecture is

1. technical definition
2. significance
3. properties
4. examples

54.1 Technical definitions

Let A be a commutative ring with identity.

Definition 54.1. An A-module is *flat* if the functor $M \otimes_A \cdot$ is exact.

For $M \otimes \cdot$ to be exact means that whenever

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence of A-modules then

$$0 \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

is exact. Note that $M \otimes \cdot$ is right exact even if M is not flat. The salient property of a flat module is that it preserves injectivity.

54.1.1 General nonsense

If A is Noetherian then M is flat if the functor $M \otimes \cdot$ preserves the exactness of any sequence of finitely generated A-modules. Even better, M is flat if for any ideal $I \subset A$ the map $M \otimes I \to M \otimes A$ is injective.

Definition 54.2. Suppose $A \to B$ is a morphism of rings. Then B is *flat over* A if B is flat as an A-module.

54.2 Examples

Example 54.3. Suppose A is a ring and S a multiplicative set. Then $S^{-1}A$ is a flat A-module, i.e., the functor $M \mapsto S^{-1}M$ is exact.

Example 54.4. If A is a Noetherian local ring then the completion $\hat{A}$ of A at its maximal ideal is flat.

Example 54.5. The *first example* of flatness was in Serre's GAGA. He called a flat module an "exact couple". The example is

$$A = \mathbf{C}[x_1, \dots, x_n] \hookrightarrow \mathbf{C}\{x_1, \dots, x_n\} \subset \hat{A}$$

in which $\mathbf{C}\{x_1, \dots, x_n\}$, the ring of convergent power series, is flat over A.

A module $B \to C$ is faithfully flat if $C \otimes_B M = 0$ implies $M = 0$. Suppose $A \to B \to C$ with C flat over A and C faithfully flat over B. Then B is flat over A. This is how Serre proved that his module was flat.

Theorem 54.6. *Suppose A, $\mathbf{m}$ is a local Noetherian ring and M is a finitely generated flat module. Then M is free.*

Proof. Take a minimal set of generators for M. Nakayama's lemma tells us how this can be done. The quotient $M/\mathbf{m}M$ is a vector space over $A/\mathbf{m}$ so it has a basis, say $\overline{m}_1, \dots, \overline{m}_r$. By Nakayama's lemma these generate M. (The module $M/(m_1, \dots, m_r)$ is sent to itself by $\mathbf{m}$ since every $x \in M$ is equivalent to an element of $(m_1, \dots, m_r)$ plus something in $\mathbf{m}M$. Thus $M/(m_1, \dots, m_r) = 0$.) This gives a surjection $A^r \to M$. Let Q be the kernel so we have an exact sequence

$$0 \to Q \to A^r \to M \to 0.$$

It is a general fact that if A is Noetherian, M is flat, and

$$0 \to R \to S \to M \to 0$$

is exact then

$$0 \to R \otimes N \to S \otimes N \to M \otimes N \to 0$$

is exact for *any* A-module N. This is proved by using the the fact that flatness implies the vanishing of $\mathrm{Tor}_1^A(M, \cdot)$.

Using the general fact tensor with $k = A/\mathbf{m}$ to obtain an exact sequence

$$0 \to Q \otimes k \to k^r \to M/\mathbf{m}M \to 0.$$

Since k^r and $M/\mathbf{m}M$ are k-vector spaces of the same dimension and the map $k^r \to M/\mathbf{m}M$ is surjective it must also be injective. Thus $Q \otimes k = 0$. But Q is finitely generated since A is Noetherian and Q is a submodule of A^r. Thus Nakayama's lemma implies $Q = 0$ so $M \cong A^r$ is free. $\square$

Theorem 54.7. *Suppose A is a d.v.r. with maximal ideal $\mathbf{m} = (t)$. Let M be any A-module. Then the following are equivalent.*

(i) M is flat,

(ii) M is torsion free,

(iii) $M \xrightarrow{t} M$ is injective.

Proof. (i)⇒(ii) Since A is a domain

$$0 \to A \xrightarrow{x} A$$

is exact for any $x \in A$. By flatness of M

$$0 \to M \xrightarrow{x} M$$

is exact so M is torsion free.

(ii)⇒(iii) is trivial

(iii)⇒(i) It is enough to check exactness for any ideal $\mathbf{a} = (t^n) \subset A$. Why is the map $M \otimes \mathbf{a} \to M \otimes A$ an isomorphism? Because under the natural identification of both modules with M the map is just multiplication by t^n which is injective by assumption. The diagram is

$$\begin{array}{ccc} M \cong M \otimes \mathbf{a} & \to & M \otimes A \cong M \\ m \mapsto m \otimes t^n & \mapsto & m \otimes t^n \mapsto t^n m \end{array}$$

□

Example 54.8. If M is a projective A module then M is flat.

54.3 Algebraic geometry definitions

Definition 54.9. Let $\mathcal{F}$ be a coherent sheaf on X and let $f : X \to Y$ be a morphism. Then $\mathcal{F}$ is *flat over* Y if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module under the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. The morphism $f : X \to Y$ is *flat* if $\mathcal{O}_X$ is flat over Y.

[["Even if f is finite, $\mathcal{F}_x$ might not be finite over $\mathcal{O}_Y$. It is only finite over the semilocal ring."]]

Example 54.10.

1. An open immersion $U \hookrightarrow X$ is flat since the local rings are the same.
2. A composition of flat morphisms is flat.
3. Base extension preserves flatness. Thus if $X \to Y$ is flat and $Y' \to Y$ then $X' = X \times_Y Y'$ is flat over Y'.

$$\begin{array}{ccc} X' = X \times_Y Y' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

   This is an exercise in local rings.

4. The product of flat morphisms is flat. [[What does this mean?]]

54.4 Families

Flatness is used for expressing the notion of a family of subschemes. For example let T be any scheme and let $X \subset \mathbf{P}^n_T$ be a closed subscheme. Then $\{X_t : t \in T\}$ is the family. Here X_t is defined by the diagram

$$\begin{array}{ccc} X_t = X \times_k \operatorname{Spec} \kappa(t) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \kappa(t) & \longrightarrow & T \end{array}$$

where $\kappa(t)$ is the residue field of $t \in T$.

This notion of family is bad since it allows for things which we do not want.

Example 54.11. Let $T = \mathbf{A}^1$ and $X \subset P^1_T = \mathbf{A}^1 \times \mathbf{P}^1$ be the union of the coordinate axis. Then for any $t \neq 0$, X_t is a point. But $X_0 = \mathbf{P}^1$.

Hard experience has lead the old pros to agree that the notion of family should require that $X \to T$ is flat. This rules out the above example since $X \to T$ is not flat. Indeed, X is defined on an open affine by $xt = 0$ so the coordinate ring is $M = k[x,t]/(xt)$. The localization of M at (x,t) has torsion as a $k[t]_{(t)}$-module. The element x is killed by t. Since $k[t]_{(t)}$ is a d.v.r. it follows that M is not a flat.

A vague generalization is the following. Suppose T is a nonsingular curve and $X \subset \mathbf{P}^n_T$. Then X is flat over T iff X has no associated point (i.e., generic points of embedded and irreducible components) whose image is a closed point of T. Equivalently one could say "every component of X dominates T." In our example the t-axis dominates T but the x-axis does not.

*Example 54.12 **(Bad)**.* Let $X \subset \mathbf{P}^1_T = \mathbf{A}^1 \times \mathbf{P}^1$ be defined by (x^2, xt). When $t \neq 0$ the fiber is $P_0 = \operatorname{Spec} k[t]$ since x localizes away. If $t = 0$ the fiber is given by the ring

$$k[x,t]/(x^2, xt) \otimes_{k[t]} k[t]/(t) \cong k[x]/(x^2)$$

which is P_0 with multiplicity 2 structure.

*Example 54.13 **(Good)**.* Let $X \subset \mathbf{P}^1_T$ be defined by $x(x-t)$. Then for $t = a \neq 0$ the fiber is $P_0 + P_a$. When $t = 0$ the fiber is $2P_0$. This is good.

Theorem 54.14. *Suppose T is a connected scheme. [[Must there be a finiteness assumption on T?]] If X is flat over T then the Hilbert polynomials $P_{X_t}(z) \in \mathbf{Q}[z]$ are independent of t. If T is integral then the converse is also true.*

Proof. Recall that for any scheme Y,

$$P_Y(m) = h^0(\mathcal{O}_Y(m)) \quad \text{for all } m \gg 0.$$

We just need to show $h^0(\mathcal{O}_{X_t}(m))$ is independent of t for all m sufficiently large. Since T is connected we reduce to the case T is affine. Furthermore we may assume $T = \operatorname{Spec} A$, with $A, \mathbf{m}$ a local Noetherian ring. [["Can get from any local ring to any other by successive specializations and generalizations. Also use the fact that flatness is preserved by base extension."]]
Claims.

1. For all $m \gg 0$, $H^i(\mathcal{O}_X(m)) = 0$ for $i > 0$ and $H^0(\mathcal{O}_X(m))$ is a free finitely generated A-module.

2. For all points $t \in T$ and all $m \gg 0$,

$$H^0(\mathcal{O}_{X_t}(m)) = H^0(\mathcal{O}_X(m)) \otimes_A \kappa(t).$$

Together these two claims imply the theorem. □

55 Theorem about flat families

Theorem 55.1. *Let $X \subset \mathbf{P}^n_T$ be a closed subscheme where T is a connected scheme of finite type over k. If X is flat over T, then the Hilbert polynomial P_{X_t} is independent of $t \in T$ (all scheme points – not just closed points!). Conversely, if T is integral and P_{X_t} is independent of t then X is flat over T.*

Proof. See Chapter III, section 9 of Hartshorne. □

56 Examples of Flat Families

*Example 56.1 (**Two lines**).* Consider the parameterized family of curves defined as follows. For each $t \in T = \operatorname{Spec} k[t]$ associate the union of the x-axis and a line parallel to the y-axis which intersects the z-axis at $z = t$. Thus for $t \neq 0$ the fiber X_t consists of the disjoint union of two lines. For $t = 0$ the fiber consists of two lines meeting at the origin.

A cohomological computation shows that the Hilbert polynomials are

$$H_{X_t}(z) = \begin{cases} 2z^2 + 1 - (-1), & \text{for } t \neq 0 \\ 2z^2 + 1, & \text{for } t = 0 \end{cases}$$

What is wrong? Let C' be the subscheme of $\mathbf{P}^3_{T-0}$ defined by the ideal

$$I_{C'} = (y, z) \cap (x, z - t) = (xy, xz, yz - tyw, z^2 - tzw).$$

(The intersection of the two ideals is the product of the given generators since they form a regular sequence.) Since the Hilbert polynomial of C' is constant over the fibers we know that C' is flat over $T - 0$. We have the following proposition.

Proposition 56.2. *Let $T = \operatorname{Spec} k[t]$ and let $X \subset \mathbf{P}^n_{T-0}$ be a closed subscheme which is flat over $T - 0$. Then there is a unique closed subscheme $\overline{X}$, flat over T, whose restriction to $\mathbf{P}^n_{T-0}$ is X. Furthermore $\overline{X}$ is the scheme-theoretic closure of X.*

Proof. See Chapter III, Section 9 of Hartshorne. □

In our situation a natural guess for $\overline{X}$ is the closed subscheme C defined by the ideal

$$(xy, xz, yz - tyw, z^2 - tzw) \subset k[t][x, y, z, w].$$

As a bonus, if we can show C is flat over T then we will know that it is the scheme-theoretic closure of C'.

To test if C is flat is suffices to show that the Hilbert polynomials are constant over the fibers. We know this when $t \neq 0$ so we need only check that the Hilbert polynomial at $t = 0$ is $2z^2 + 2$. At $t = 0$ the defining ideal becomes

$$(xy, xz, yz, z^2) = (x, z) \cap (y, z) \cap (x, y, z)^2.$$

This looks like the union of the x and y with an embedded point at the origin. The arithmetic genus of the degree $d = 2$ plane curve defined by (z, xy) is

$$p_a = \frac{1}{2}(d-1)(d-2) = 0$$

so it has Hilbert polynomial $2z + 1$. The ith graded piece of $k[x, y, z, w]/(xy, xz, yz, z^2)$ has dimension one more than the dimension of the ith graded piece of $k[x, y, z, w]/(z, xy)$. (In degree i the first ring has $w^{i-1}z$ whereas the latter does not.) The Hilbert polynomial is then

$$P_{C_0}(z) = (2z + 1) + 1 = 2z + 2.$$

Thus $P_{C_t}(z) = 2z + 2$ for all t so by the big theorem from the last lecture $C \subset \mathbf{P}^3_T$ is a flat family.

Note that although the Hilbert polynomial is constant along the fibers the cohomology of C_0 is different than that of the other fibers. This family can sort of be thought of as 2 planes meeting at a point in 4-dimensional space.

> "This is an example of the genius of Grothendieck. The Italians knew that when two planes came together the genus changed but they were not able to deal with it like Grothendieck did."

Example 56.3 ***(The Twisted Cubic)***. This is the last example of the course. Some say this is the only example in algebraic geometry. Let $T = \operatorname{Spec} k[t]$. The fiber over 1 in our family will be $C_1 \subset \mathbf{P}^3_k$ defined by

$$I_{C_1} = (xy - zw, x^2 - yw, y^2 - xz).$$

How can C_1 give rise to a family? One way is to assign weights to the variables then homogenize with respect to a new variable t. Assign weights as in the following table.

variable	x	y	z	w
weight	8	4	2	1

Introducing a new variable t of weight 1 and homogenizing we obtain the ideal

$$I = (xy - t^9 zw, x^2 - t^{11} yw, xz - t^2 y^2) \subset k[t][x, y, z, w].$$

This ideal defines a closed subscheme C of $\mathbf{P}^3_T$ since I is homogeneous as an ideal in the polynomial ring $k[t][x, y, z, w]$.

Is C a flat family? When $t \neq 0$, C_t is a curve. But C_0 is defined by the ideal

$$(xy, x^2, xz) = (x) \cap (x, y, z)^2$$

so it is the entire y–z plane plus an embedded point. This is way too large so C is not a flat family. There must be some torsion. Indeed,

$$xyz - t^9 z^2 w, xyz - t^2 y^3 \in I$$

so

$$t^2(y^3 - t^7 z^2 w) = t^2 y^3 - t^9 z^2 w \in I.$$

Thus $y^3 - t^7 z^2 w$ is a torsion element of $k[t][x, y, z, w]/I$ over $k[t]$. So now add $y^3 - t^7 z^2 w$ to I. Let C' be the closed subscheme defined by the ideal

$$I' = (y^3 - t^7 z^2 w, xz - t^2 y^2, x^2 - t'' yw, xy - t^9 zw).$$

When $t = 0$ the fiber C'_0 is defined by

$$(xy, x^2, xz, y^3) = (x, y^3) \cap (x^2, xy, xz, z^2, y^3).$$

One can show C'_0 is of degree 3 and has arithmetic genus 0. Thus C' is a flat family. The basis we have given for I' is called a *Gröbner basis.*

[[At this point the class came to an end. Everyone clapped, then clapped some more. It is clear that we really appreciated Hartshorne. He did a great job.]]

57 Homework problems

I did not type up solutions to the first homework set. I did type of solutions to the other homework sets.

57.1 Exercise on basic cohomology and abstract nonsense

Exercise 57.1. For each of the following categories, decide whether the category has enough projective objects (i.e. every object is a quotient of a projective object).

a) $\mathbf{Ab}(X)$, where X is a topological space,

b) $\mathbf{Mod}(X)$, where $(X, \mathcal{O}_X)$ is a ringed space,

c) $\mathbf{Qco}(X)$, where X is a Noetherian scheme,

d) $\mathbf{Qco}(X)$, where X is a Noetherian affine scheme.

Exercise 57.2. Let $X = \mathbf{P}^1_k$, over an algebraically closed field k. Using only material from ChIII, §1,2 (esp. Exercise 2.2) show

a) For any coherent sheaf $\mathcal{F}$ on X,

$$H^i(X, \mathcal{F}) = 0 \quad \text{for all } i \geq 2.$$

[Hint: Treat the case $\mathcal{F}$ torsion and $\mathcal{F}$ torsion-free separately; in the torsion-free case, tensor the exact sequence of Exercise 2.2 by $\mathcal{F}$.]

b) Show that for all $\ell > 0$, $H^1(X, \mathcal{O}_X(\ell)) = 0$.

Exercise 57.3. Let X be an integral Noetherian scheme.

a) Show that the sheaf $\mathcal{K}^*$ (cf. II §6) is flasque. Conclude that $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*)$.

b) Give an example of a Noetherian affine scheme with $H^1(X, \mathcal{O}_X^*) \neq 0$.

Exercise 57.4. Let X be a Noetherian scheme.

a) Show that the sheaf $\mathcal{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathbf{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathbf{Qco}(X)$ has enough injectives.

b) Show that any injective object of $\mathbf{Qco}(X)$ is flasque. [*Hints:* The method of proof of (2.4) will *not* work, because $\mathcal{O}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathcal{I} \in \mathbf{Qco}(X)$ is injective, and if $U \subset X$ is an open subset, then $\mathcal{I}|_U$ is an injective object of $\mathbf{Qco}(U)$. Then cover X with open affine...]

c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathbf{Qco}(X)$ to $\mathbf{Ab}$.

57.2 Chapter III, 4.8, 4.9, 5.6

57.2.1 Exercise III.4.8: Cohomological Dimension

Let X be a Noetherian separated scheme. We define the cohomological dimension *of X, denoted* $\mathrm{cd}(X)$, *to be the least integer n such that $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that* $\mathrm{cd}(X) = 0$ *if and only if X is affine. Grothendieck's theorem (2.7) implies that* $\mathrm{cd}(X) \leq \dim X$.

(a) In the definition of $\mathrm{cd}(X)$, *show that it is sufficient to consider only coherent sheaves on X.*

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X.

(c) Suppose X has a covering by $r+1$ open affine subsets. Use Čech cohomology to show that $\mathrm{cd}(X) \leq r$.

(d) If X is a quasi-projective variety of dimension r over a field k, then X can be covered by $r+1$ open affine subsets. Conclude that $\mathrm{cd}(X) \leq \dim X$.

(e) Let Y be a set-theoretic complete intersection of codimension r in $X = \mathbf{P}^n_k$. Show that $\mathrm{cd}(X - Y) \leq r - 1$.

Proof. (a) It suffices to show that if, for some i, $H^i(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$, then $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$. Thus suppose the ith cohomology of all coherent sheaves on X vanishes and let $\mathcal{F}$ be quasi-coherent. Let $(\mathcal{F}_\alpha)$ be the collection of coherent subsheaves of $\mathcal{F}$, ordered by inclusion. Then by (II, Ex. 5.15e) $\varinjlim \mathcal{F}_\alpha = \mathcal{F}$, so by (2.9)

$$H^i(X, \mathcal{F}) = H^i(X, \varinjlim \mathcal{F}_\alpha) = \varinjlim H^i(X, \mathcal{F}_\alpha) = 0.$$

(b) Suppose n is an integer and $H^i(X, \mathcal{F}) = 0$ for all coherent locally free sheaves $\mathcal{F}$ and integers $i > n$. We must show $H^i(X, \mathcal{F}) = 0$ for all coherent $\mathcal{F}$ and all $i > n$, then applying (a) gives the desired result. Since X is quasiprojective there is an open immersion

$$i : X \hookrightarrow Y \subset \mathbf{P}^n_k$$

with Y a closed subscheme of $\mathbf{P}^n_k$ and $i(X)$ open in Y. By (II, Ex. 5.5c) the sheaf $\mathcal{F}$ on X pushes forward to a coherent sheaf on $\mathcal{F}' = i_*\mathcal{F}$ on Y. By (II, 5.18) we may write $\mathcal{F}'$ as a quotient of a locally free coherent sheaf $\mathcal{E}'$ on Y. Letting $\mathbf{R}'$ be the kernel gives an exact sequence

$$0 \to \mathbf{R}' \to \mathcal{E}' \to \mathcal{F}' \to 0$$

with R' coherent (it's the quotient of coherent sheaves). Pulling back via i to X gives an exact sequence

$$0 \to \mathbf{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

of coherent sheaves on X with E locally free. The long exact sequence of cohomology shows that for $i > n$, there is an exact sequence

$$0 = H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathbf{R}) \to H^{i+1}(X, \mathcal{E}) = 0.$$

$H^i(X, \mathcal{E}) = H^{i+1}(X, \mathcal{E}) = 0$ because we have assumed that, for $i > n$, cohomology vanishes on locally free coherent sheaves. Thus $H^i(X, \mathcal{F}) \cong H^{i+1}(X, \mathbf{R})$. But if $k = \dim X$, then Grothendieck vanishing (2.7) implies that $H^{k+1}(X, \mathbf{R}) = 0$ whence $H^k(X, \mathcal{F}) = 0$. But then applying the above argument with $\mathcal{F}$ replaced by $\mathbf{R}$ shows that $H^k(X, \mathbf{R}) = 0$ which implies $H^{k-1}(X, \mathcal{F}) = 0$ (so long as $k - 1 > n$). Again, apply the entire argument with $\mathcal{F}$

replaced by $\mathbf{R}$ to see that $H^{k-1}(X, \mathbf{R}) = 0$. We can continue this descent and hence show that $H^i(X, \mathcal{F}) = 0$ for all $i > n$.

(c) By (4.5) we can compute cohomology by using the Čech complex resulting from the cover $\mathfrak{U}$ of X by $r+1$ open affines. By definition $\mathcal{C}^p = 0$ for all $p > r$ since there are no intersections of $p+1 \geq r+2$ distinct open sets in our collection of $r+1$ open sets. The Čech complex is

$$\mathcal{C}^0 \to \mathcal{C}^1 \to \cdots \to \mathcal{C}^r \to \mathcal{C}^{r+1} = 0 \to 0 \to 0 \to \cdots .$$

Thus if $\mathcal{F}$ is quasicoherent then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = 0$ for any $p > r$ which implies that $\mathrm{cd}(X) \leq r$.

(d) I will first present my solution in the special case that X is projective. The more general case when X is quasi-projective is similar, but more complicated, and will be presented next. Suppose $X \subset \mathbf{P}^n$ is a projective variety of dimension r. We must cover X with $r+1$ open affines. Let U be nonempty open affine subset of X. Since X is irreducible, the irreducible components of $X - U$ all have codimension at least one in X. Now pick a hyperplane H which doesn't completely contain any irreducible component of $X - U$. We can do this by choosing one point P_i in each of the finitely many irreducible components of $X - U$ and choosing a hyperplane which avoids all the P_i. This can be done because the field is infinite (varieties are only defined over algebraically closed fields) so we can always choose a vector not orthogonal to any of a finite set of vectors. Since X is closed in $\mathbf{P}^n$ and $\mathbf{P}^n - H$ is affine, $(\mathbf{P}^n - H) \cap X$ is an open affine subset of X. Because of our choice of H, $U \cup ((\mathbf{P}^n - H) \cap X)$ is only missing codimension two closed subsets of X. Let $H_1 = H$ and choose another hyperplane H_2 so it doesn't completely contain any of the (codimension two) irreducible components of $X - U - (\mathbf{P}^n - H_1)$. Then $(\mathbf{P}^n - H_2) \cap X$ is open affine and $U \cup ((\mathbf{P}^n - H_1) \cap X) \cup ((\mathbf{P}^n - H2) \cap X)$ is only missing codimension three closed subsets of X. Repeating this process a few more times yields hyperplanes $H_1, \cdots, H_r$ so that

$$U, (\mathbf{P}^n - H_1) \cap X, \ldots, (\mathbf{P}^n - H_r) \cap X$$

form an open affine cover of X, as desired.

Now for the quasi-projective case. Suppose $X \subset \mathbf{P}^n$ is quasi-projective. From (I, Ex. 3.5) we know that $\mathbf{P}^n$ minus a hypersurface H is affine. Note that the same proof works even if H is a union of hypersurfaces. We now proceed with the same sort of construction as in the projective case, but we must choose H more cleverly to insure that $(\mathbf{P}^n - H) \cap X$ is affine. Let U be a nonempty affine open subset of X. As before pick a hyperplane which doesn't completely contain any irreducible component of $X - U$. Since X is only quasi-projective we can't conclude that $(\mathbf{P}^n - H) \cap X$ is affine. But we do know that $(\mathbf{P}^n - H) \cap \overline{X}$ is affine. Our strategy is to add some hypersurfaces to H to get a union of hypersurfaces S so that

$$(\mathbf{P}^n - S) \cap \overline{X} = (\mathbf{P}^n - S) \cap X.$$

But, we must be careful to add these hypersurfaces in such a way that $((\mathbf{P}^n - S) \cap X) \cup U$ is missing only codimension two or greater subsets of X. We do this as follows. For each irreducible component Y of $\overline{X} - X$ choose a hypersurface H' which completely contains Y but which does not completely contain any irreducible component of $X - U$. That this can be done is the content of a lemma which will be proved later (just pick a point in each irreducible component and avoid it). Let S by the union of all of the H' along with H. Then $\mathbf{P}^n - S$ is affine and so

$$(\mathbf{P}^n - S) \cap X = (\mathbf{P}^n - S) \cap \overline{X}$$

is affine. Furthermore, S properly intersects all irreducible components of $X - U$, so $((\mathbf{P}^n - S) \cap X) \cup U$ is missing only codimension two or greater subsets of X. Repeating this process as above several times yields the desired result because after each repetition the codimension of the resulting pieces is reduced by 1.

Lemma 57.5. *If Y is a projective variety and $p_1, \dots, p_n$ is a finite collection of points not on Y, then there exists a (possibly reducible) hypersurface H containing Y but not containing any of the p_i.*

By a possibly reducible hypersurface I mean a union of irreducible hypersurfaces, not a hypersurface union higher codimension varieties.

Proof. This is obviously true and I have a proof, but I think there is probably a more algebraic proof. Note that k is infinite since we only talk about varieties over algebraically closed fields. Let $f_1, \cdots, f_m$ be defining equations for Y. Thus Y is the common zero locus of the f_i and not all f_i vanish on any p_i. I claim that we can find a linear combination $\sum a_i f_i$ of the f_i which doesn't vanish on any p_i. Since k is infinite and not all f_i vanish on p_1, we can easily find a_i so that $\sum a_i f_i(p_1) \neq 0$ and all the $a_i \neq 0$. If $\sum a_i f_i(p_2) = 0$ then, once again since k is infinite, we can easily "jiggle" the a_i so that $\sum a_i f_i(p_2) \neq 0$ and $\sum a_i f_i(p_1)$ is still nonzero. Repeating this same argument for each of the finitely many points p_i gives a polynomial $f = \sum a_i f_i$ which doesn't vanish on any p_i. Of course I want to use f to define our hypersurface, but I can't because f might not be homogeneous. Fortunately, this is easily dealt with by suitably multiplying the various f_i by the defining equation of a hyperplane not passing through any p_i, then repeating the above argument. Now let H be the hypersurface defined by $f = \sum a_i f_i$. Then by construction H contains Y and H doesn't contain any p_i. □

(e) Suppose Y is a set-theoretic complete intersection of codimension r in $X = \mathbf{P}^n_k$. Then Y is the intersection of r hypersurfaces, so we can write $Y = H_1 \cap \cdots \cap H_r$ where each H_i is a hypersurface. By (I, Ex. 3.5) $X - H_i$ is affine for each i, thus

$$X - Y = (X - H_1) \cup \cdots \cup (X - H_r)$$

can be covered by r open affine subsets. By (c) this implies $\mathrm{cd}(X - Y) \leq r - 1$ which completes the proof. □

57.2.2 Exercise III.4.9

Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y}$ in $\mathbf{P}^4_k$ is not a set-theoretic complete intersection.

Proof. By (Ex. 4.8e) it suffices to show that $H^2(X - Y, \mathcal{O}_{X-Y}) \neq 0$. Suppose Z is a closed subset of X, then by (Ex. 2.3d), for any $i \geq 1$, there is an exact sequence

$$H^i(X, \mathcal{O}_X) \to H^i(X - Z, \mathcal{O}_{X-Z}) \to H^{i+1}_Z(X, \mathcal{O}_X) \to H^{i+1}(X, \mathcal{O}_X).$$

By (3.8), $H^i(X, \mathcal{O}_X) = H^{i+1}(X, \mathcal{O}_X) = 0$ so $H^i(X - Z, \mathcal{O}_{X-Z}) = H^{i+1}_Z(X, \mathcal{O}_X)$. Applying this with $Z = Y$ and $i = 2$ shows that

$$H^2(X - Y, \mathcal{O}_{X-Y}) = H^3_Y(X, \mathcal{O}_X).$$

Thus we just need to show that $H^3_Y(X, \mathcal{O}_X) \neq 0$.

Mayer-Vietoris (Ex. 2.4) yields an exact sequence

$$H^3_{Y_1}(X, \mathcal{O}_X) \oplus H^3_{Y_2}(X, \mathcal{O}_X) \to H^3_Y(X, \mathcal{O}_X) \to$$
$$H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \to H^4_{Y_1}(X, \mathcal{O}_X) \oplus H^4_{Y_2}(X, \mathcal{O}_X)$$

As above, $H^3_{Y_1}(X, \mathcal{O}_X) = H^2(X - Y_1, \mathcal{O}_{X-Y_1})$. But $X - Y_1$ is a set-theoretic complete intersection of codimension 2 so $\operatorname{cd}(X - Y_1) \leq 1$, whence $H^2(X - Y_1, \mathcal{O}_{X-Y_1}) = 0$. Similarly

$$H^2(X - Y_2, \mathcal{O}_{X-Y_2}) = H^3(X - Y_1, \mathcal{O}_{X-Y_1}) = H^3(X - Y_2, \mathcal{O}_{X-Y_2}) = 0.$$

Thus from the above exact sequence we see that $H^3_Y(X, \mathcal{O}_X) = H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X)$.

Let $P = Y_1 \cap Y_2 = \{(0,0,0,0)\}$. We have reduced to showing that $H^4_P(X, \mathcal{O}_X)$ is nonzero. Since $H^4_P(X, \mathcal{O}_X) = H^3(X - P, \mathcal{O}_{X-P})$ we can do this by a direct computation of $H^3(X - P, \mathcal{O}_{X-P})$ using Čech cohomology. Cover $X - P$ by the affine open sets $U_i = \{x_i \neq 0\}$. Then the Čech complex is

$$\begin{aligned}
&k[x_1,x_2,x_3,x_4,x_1^{-1}] \oplus \cdots \oplus k[x_1,x_2,x_3,x_4,x_4^{-1}] \xrightarrow{d_0} \\
&k[x_1,x_2,x_3,x_4,x_1^{-1},x_2^{-1}] \oplus \cdots \oplus k[x_1,x_2,x_3,x_4,x_3^{-1},x_4^{-1}] \xrightarrow{d_1} \\
&k[x_1,x_2,x_3,x_4,x_1^{-1},x_2^{-1},x_3^{-1}] \oplus \cdots \oplus k[x_1,x_2,x_3,x_4,x_2^{-1},x_3^{-1},x_4^{-1}] \xrightarrow{d_2} \\
&k[x_1,x_2,x_3,x_4,x_1^{-1},x_2^{-1},x_3^{-1},x_4^{-1}]
\end{aligned}$$

Thus

$$H^3(X - P, \mathcal{O}_{X-P}) = \{x_1^i x_2^j x_3^k x_4^\ell : i,j,k,\ell < 0\} \neq 0.$$

$\square$

57.2.3 Exercise III.5.6: Curves on a nonsingular quadric surface

Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}^3_k$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\operatorname{Pic} Q \cong \mathbf{Z} \oplus \mathbf{Z}$, *so we can talk about the* type *(a,b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a,b)$. Thus for any $n \in \mathbf{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n,n)$.*

*[***Comment!*** In my solution, a subscheme Y of type (a,b) corresponds to the invertible sheaf $\mathcal{O}_Q(-a,-b)$. I think this is reasonable since then $\mathcal{O}_Q(-a,-b) = \mathcal{L}(-Y) = \mathcal{I}_Y$. The correspondence is not clearly stated in the problem, but this choice works.]*

(a) Use the special case $(q,0)$ and $(0,q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q, to show:

1. *if $|a - b| \leq 1$, then $H^1(Q, \mathcal{O}_Q(a,b)) = 0$;*
2. *if $a, b < 0$, then $H^1(Q, \mathcal{O}_Q(a,b)) = 0$;*
3. *if $a \leq -2$, then $H^1(Q, \mathcal{O}_Q(a,0)) \neq 0$).*

Solution. First I will prove a big lemma in which I explicitly calculate $H^1(Q, \mathcal{O}_Q(0,-q))$ and some other things which will come in useful later. Next I give an independent computation of the other cohomology groups (1), (2).

Lemma 57.6. *Let $q > 0$, then*

$$\dim_k H^1(Q, \mathcal{O}_Q(-q,0)) = H^1(Q, \mathcal{O}_Q(0,-q)) = q - 1.$$

Furthermore, we know all terms in the long exact sequence of cohomology associated with the short exact sequence

$$0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Proof. We prove the lemma only for $\mathcal{O}_Q(-q,0)$, since the argument for $\mathcal{O}_Q(0,-q)$ is exactly the same. Suppose Y is the disjoint union of q lines $\mathbf{P}^1$ in Q so $\mathcal{I}_Y = \mathcal{O}_Q(-q,0)$. The sequence

$$0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

is exact. The associated long exact sequence of cohomology is

$$\begin{aligned} 0 \to &\Gamma(Q,\mathcal{O}_Q(-q,0)) \to \Gamma(Q,\mathcal{O}_Q) \to \Gamma(Q,\mathcal{O}_Y) \\ \to &H^1(Q,\mathcal{O}_Q(-q,0)) \to H^1(Q,\mathcal{O}_Q) \to H^1(Q,\mathcal{O}_Y) \\ \to &H^2(Q,\mathcal{O}_Q(-q,0)) \to H^2(Q,\mathcal{O}_Q) \to H^2(Q,\mathcal{O}_Y) \to 0 \end{aligned}$$

We can compute all of the terms in this long exact sequence. For the purposes at hand it suffices to view the summands as k-vector spaces so we systematically do this throughout. Since $\mathcal{O}_Q(-q,0) = \mathcal{I}_Y$ is the ideal sheaf of Y, its global sections must vanish on Y. But $\mathcal{I}_Y$ is a subsheaf of $\mathcal{O}_Q$ whose global sections are the constants. Since the only constant which vanish on Y is 0, $\Gamma(Q,\mathcal{O}_Q(-q,0)) = 0$. By (I, 3.4), $\Gamma(Q,\mathcal{O}_Q) = k$. Since Y is the disjoint union of q copies of $\mathbf{P}^1$ and each copy has global sections k, $\Gamma(Q,\mathcal{O}_Y) = k^{\oplus q}$. Since Q is a complete intersection of dimension 2, (Ex. 5.5 b) implies $H^1(Q,\mathcal{O}_Q) = 0$. Because Y is isomorphic to several copies of $\mathbf{P}^1$, the general result (proved in class, but not in the book) that $H^n_*(\mathcal{O}_{\mathbf{P}^n}) = \{\sum a_I X_I : \text{entries in } I \text{ negative}\}$ implies $H^1(Q,\mathcal{O}_Y) = H^1(Y,\mathcal{O}_Y) = 0$. Since Q is a hypersurface of degree 2 in $\mathbf{P}^3$, (I, Ex. 7.2(c)) implies $p_a(Q) = 0$. Thus by (Ex. 5.5c) we see that $H^2(Q,\mathcal{O}_Q) = 0$. Putting together the above facts and some basic properties of exact sequences show that $H^1(Q,\mathcal{O}_Q(-q,0)) = k^{\oplus(q-1)}$, $H^2(Q,\mathcal{O}_Q(-q,0)) = 0$ and $H^2(Q,\mathcal{O}_Y) = 0$. Our long exact sequence is now

$$\begin{aligned} 0 \to &\Gamma(Q,\mathcal{O}_Q(-q,0)) = 0 \to \Gamma(Q,\mathcal{O}_Q) = k \to \Gamma(Q,\mathcal{O}_Y) = k^{\oplus q} \\ \to &H^1(Q,\mathcal{O}_Q(-q,0)) = k^{\oplus(q-1)} \to H^1(Q,\mathcal{O}_Q) = 0 \to H^1(Q,\mathcal{O}_Y) = 0 \\ \to &H^2(Q,\mathcal{O}_Q(-q,0)) = 0 \to H^2(Q,\mathcal{O}_Q) = 0 \to H^2(Q,\mathcal{O}_Y) = 0 \to 0 \end{aligned}$$

$\square$

Number (3) now follows immediately from the lemma because

$$H^1(Q,\mathcal{O}_Q(a,0)) = k^{\oplus(-a-1)} \neq 0$$

for $a \leq -2$.

Now we compute (1) and (2). Let a be an arbitrary integer. First we show that $\mathcal{O}_Q(a,a) = 0$. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-2) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_Q \to 0$$

where the first map is multiplication by $xy - zw$. Twisting by a gives an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-2+a) \to \mathcal{O}_{\mathbf{P}^3}(a) \to \mathcal{O}_Q(a) \to 0.$$

The long exact sequence of cohomology yields an exact sequence

$$\cdots \to H^1(\mathcal{O}_{\mathbf{P}^3}(a)) \to H^1(\mathcal{O}_Q(a)) \to H^2(\mathcal{O}_{\mathbf{P}^3}(-2+a)) \to \cdots$$

But from the explicit computations of projective space (5.1) it follows that $H^1(\mathcal{O}_{\mathbf{P}^3}(a)) = 0$ and $H^2(\mathcal{O}_{\mathbf{P}^3}(-2+a)) = 0$ from which we conclude that $H^1(\mathcal{O}_Q(a)) = 0$.

Next we show that $\mathcal{O}_Q(a-1,a)=0$. Let Y be a single copy of $\mathbf{P}^1$ sitting in Q so that Y has type $(1,0)$. Then we have an exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

But $\mathcal{I}_Y = \mathcal{O}_Q(-1,0)$ so this becomes

$$0 \to \mathcal{O}_Q(-1,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Now twisting by a yields the exact sequence

$$0 \to \mathcal{O}_Q(a-1,a) \to \mathcal{O}_Q(a) \to \mathcal{O}_Y(a) \to 0.$$

The long exact sequence of cohomology gives an exact sequence

$$\cdots \to \Gamma(\mathcal{O}_Q(a)) \to \Gamma(\mathcal{O}_Y(a)) \to H^1(\mathcal{O}_Q(a-1,a)) \to H^1(\mathcal{O}_Q(a)) \to \cdots$$

We just showed that $H^1(\mathcal{O}_Q(a)) = 0$, so to see that $H^1(\mathcal{O}_Q(a-1,a)) = 0$ it suffices to note that the map $\Gamma(\mathcal{O}_Q(a)) \to \Gamma(\mathcal{O}_Y(a))$ is surjective. This can be seen by writing $Q = \operatorname{Proj}(k[x,y,z,w]/(xy-zw))$ and (w.l.o.g.) $Y = \operatorname{Proj}(k[x,y,z,w]/(xy-zw,x,z))$ and noting that the degree a part of $k[x,y,z,w]/(xy-zw)$ surjects onto the degree a part of $k[x,y,z,w]/(xy-zw,x,z)$. Thus $H^1(\mathcal{O}_Q(a-1,a)) = 0$ and exactly the same argument shows $H^1(\mathcal{O}_Q(a,a-1)) = 0$. This gives (1).

For (2) it suffices to show that for $a>0$,

$$H^1(\mathcal{O}_Q(-a,-a-n)) = H^1(\mathcal{O}_Q(-a-n,-a)) = 0$$

for all $n>0$. Thus let $n>0$ and suppose Y is a disjoint union of n copies of $\mathbf{P}^1$ in such a way that $\mathcal{I}_Y = \mathcal{O}_Q(0,-n)$. Then we have an exact sequence

$$0 \to \mathcal{O}_Q(0,-n) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Twisting by $-a$ yields the exact sequence

$$0 \to \mathcal{O}_Q(-a,-a-n) \to \mathcal{O}_Q(-a) \to \mathcal{O}_Y(-a) \to 0.$$

The long exact sequence of cohomology then gives an exact sequence

$$\cdots \to \Gamma(\mathcal{O}_Y(-a)) \to H^1(\mathcal{O}_Q(-a,-a-n)) \to H^1(\mathcal{O}_Q(-a)) \to \cdots$$

As everyone knows, since Y is just several copies of $\mathbf{P}^1$ and $-a<0$, $\Gamma(\mathcal{O}_Y(-a)) = 0$. Because of our computations above, $H^1(\mathcal{O}_Q(-a)) = 0$. Thus $H^1(\mathcal{O}_Q(-a,-a-n)) = 0$, as desired. Showing that $H^1(\mathcal{O}_Q(-a-n,-a)) = 0$ is exactly the same.

(b) Now use these results to show:

1. If Y is a locally principal closed subscheme of type (a,b) with $a,b>0$, then Y is connected.

   Proof. Computing the long exact sequence associated to the short exact sequence

   $$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

   gives the exact sequence

   $$0 \to \Gamma(Q,\mathcal{I}_Y) \to \Gamma(Q,\mathcal{O}_Q) \to \Gamma(Q,\mathcal{O}_Y) \to H^1(Q,\mathcal{I}_Y) \to \cdots$$

   But, $\Gamma(\mathcal{I}_Y) = 0$, $\Gamma(Q,\mathcal{O}_Q) = k$, and by (a)2 above $H^1(Q,\mathcal{I}_Y) = H^1(Q,\mathcal{O}_Q(-a,-b)) = 0$. Thus we have an exact sequence

   $$0 \to 0 \to k \to \Gamma(\mathcal{O}_Y) \to 0 \to \cdots$$

   from which we conclude that $\Gamma(\mathcal{O}_Y) = k$ which implies Y is connected. $\square$

2. now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a, b). Use (II, 7.6.2) and (II, 8.18).

Proof. Given (a, b), (II, 7.6.2) gives a closed immersion

$$Q = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^a \times \mathbf{P}^b \to \mathbf{P}^n$$

which corresponds to the invertible sheaf $\mathcal{O}_Q(-a, -b)$ of type (a, b). By Bertini's theorem (II, 8.18) there is a hyperplane H in $\mathbf{P}^n$ such that the hyperplane section of the (a, b) embedding of Q in $\mathbf{P}^n$ is nonsingular. Pull this hyperplane section back to a nonsingular curve Y of type (a, b) on Q in $\mathbf{P}^3$. By the previous problem, Y is connected. Since Y comes from a hyperplane section this implies Y is irreducible (see the remark in the statement of Bertini's theorem). □

3. an irreducible nonsingular curve Y of type (a, b), $a, b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a - b| \leq 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type $(1, 3)$ which is just the rational quartic curve (I, Ex. 3.18).

Proof. Let Y be an irreducible nonsingular curve of type (a, b). The criterion we apply comes from (II, Ex 5.14d) which asserts that the maps

$$\Gamma(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$$

are surjective for all $n \geq 0$ if and only if Y is projectively normal. To determine when this occurs we have to replace $\Gamma(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n))$ with $\Gamma(Q, \mathcal{O}_Q(n))$. It is easy to see that the above criterion implies we can make this replacement if Q is projectively normal. Since $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$ is locally isomorphic to $\mathbf{A}^1 \times \mathbf{A}^1 \cong \mathbf{A}^2$ which is normal, we see that Q is normal. Then since Q is a complete intersection which is normal, (II, 8.4b) implies Q is projectively normal.

Consider the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y.$$

Twisting by n gives an exact sequence

$$0 \to \mathcal{I}_Y(n) \to \mathcal{O}_Q(n) \to \mathcal{O}_Y(n).$$

Taking cohomology yields the exact sequence

$$\cdots \to \Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Q, \mathcal{O}_Y(n)) \to H^1(Q, \mathcal{I}_Y(n)) \to \cdots$$

Thus Y is projectively normal precisely if $H^1(Q, \mathcal{I}_Y(n)) = 0$ for all $n \geq 0$. When can this happen? We apply our computations from part (a). Since $\mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$,

$$\mathcal{I}_Y(n) = \mathcal{O}_Q(-a, -b)(n) = \mathcal{O}_Q(-a, -b) \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(n, n) = \mathcal{O}_Q(n - a, n - b)$$

If $|a - b| \leq 1$ then $|(n - a) - (n - b)| \leq 1$ for all n so

$$H^1(Q, \mathcal{O}_Q(-a, -b)(n)) = 0$$

for all n which implies Y is projectively normal. On the other hand, if $|a - b| > 1$ let n be the minimum of a and b, without loss assume b is the minimum, so $n = b$. Then from (a) we see that

$$\mathcal{O}_Q(-a, -b)(n) = \mathcal{O}_Q(-a, -b)(b) = \mathcal{O}_Q(-a + b, 0) \neq 0$$

since $-a + b \leq -2$. □

(c) If Y is a locally principal subscheme of type (a, b) in Q, show that $p_a(Y) = ab-a-b+1$. [Hint: Calculate the Hilbert polynomials of suitable sheaves, and again use the special case (q,0) which is a disjoint union of q copies of $\mathbf{P}^1$.]

Proof. The sequence

$$0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

is exact so

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Q) - \chi(\mathcal{O}_Q(-a, -b)) = 1 - \chi(\mathcal{O}_Q(-a, -b)).$$

Thus

$$p_a(Y) = 1 - \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Q(-a, -b)).$$

The problem is thus reduced to computing $\chi(\mathcal{O}_Q(-a, -b))$.

Assume first that $a, b < 0$. To compute $\chi(\mathcal{O}_Q(-a, -b))$ assume $Y = Y_1 \cup Y_2$ where $\mathcal{I}_{Y_1} = \mathcal{O}_Q(-a, 0)$ and $\mathcal{I}_{Y_2} = \mathcal{O}_Q(0, -b)$. Thus we could take Y_1 to be a copies of $\mathbf{P}^1$ in one family of lines and Y_2 to be b copies of $\mathbf{P}^1$ in the other family. Tensoring the exact sequence

$$0 \to \mathcal{I}_{Y_1} \to \mathcal{O}_Q \to \mathcal{O}_{Y_1} \to 0$$

by the flat module $\mathcal{I}_{Y_2}$ yields an exact sequence

$$0 \to \mathcal{I}_{Y_1} \otimes \mathcal{I}_{Y_2} \to \mathcal{I}_{Y_2} \to \mathcal{O}_{Y_1} \otimes \mathcal{I}_{Y_2}$$

[Note: I use the fact that $\mathcal{I}_{Y_2}$ is flat. This follows from a proposition in section 9 which we haven't yet reached, but I'm going to use it anyways. Since Y_2 is locally principal, $\mathcal{I}_{Y_2}$ is generated locally by a single element and since Q is a variety it is integral. Thus $\mathcal{I}_{Y_2}$ is locally free so by (9.2) $\mathcal{I}_{Y_2}$ is flat.] This exact sequence can also be written as

$$0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q(0, -b) \to \mathcal{O}_Y \otimes \mathcal{O}_Q(0, -b) \to 0.$$

The associated long exact sequence of cohomology is

$$\begin{aligned}
0 \to &\Gamma(Q, \mathcal{O}_Q(-a, -b)) \to \Gamma(Q, \mathcal{O}_Q(0, -b)) \to \Gamma(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0, -b)) \\
\to &H^1(Q, \mathcal{O}_Q(-a, -b)) \to H^1(Q, \mathcal{O}_Q(0, -b)) \to H^1(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0, -b)) \\
\to &H^2(Q, \mathcal{O}_Q(-a, -b)) \to H^2(Q, \mathcal{O}_Q(0, -b)) \to H^2(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0, -b)) \to 0
\end{aligned}$$

The first three groups of global sections are 0. Since $a, b < 0$, (a) implies $H^1(Q, \mathcal{O}_Q(-a, -b)) = 0$. From the lemma we know that $H^1(Q, \mathcal{O}_Q(0, -b)) = k^{\oplus(b-1)}$. Also by the lemma we know that $H^2(Q, \mathcal{O}_Q(0, -b)) = 0$. Since $\mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0, -b)$ is isomorphic to the ideal sheaf of $b-1$ points in each line of Y_1, a similar proof as that used in the lemma shows that

$$H^1(Q, \mathcal{O}_Y \otimes \mathcal{O}_Q(0, -b)) = k^{\oplus a(b-1)}.$$

Plugging all of this information back in yields the exact sequence

$$\begin{aligned}
0 \to &\Gamma(Q, \mathcal{O}_Q(-a, -b)) = 0 \to \Gamma(Q, \mathcal{O}_Q(0, -b)) = 0 \to \Gamma(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0, -b)) = 0 \\
\to &H^1(Q, \mathcal{O}_Q(-a, -b)) = 0 \to H^1(Q, \mathcal{O}_Q(0, -b)) = k^{\oplus(b-1)} \\
&\qquad \to H^1(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0, -b)) = k^{\oplus a(b-1)} \\
\to &H^2(Q, \mathcal{O}_Q(-a, -b)) \to H^2(Q, \mathcal{O}_Q(0, -b)) = 0 \\
&\qquad \to H^2(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0, -b)) = 0 \to 0
\end{aligned}$$

From this we conclude that

$$\chi(\mathcal{O}_Q(-a,-b)) = 0 + 0 + h^2(Q, \mathcal{O}_Q(-a,-b)) = a(b-1) - (b-1) = ab - a - b + 1$$

which is the desired result.

Now we deal with the remaining case, when Y is a disjoint copies of $\mathbf{P}^1$. We have

$$p_a(Y) = 1 - \chi(\mathcal{O}_Y) = 1 - \chi(\mathcal{O}_{\mathbf{P}^1}^{\oplus a}) = 1 - a\chi(\mathcal{O}_{\mathbf{P}^1}) = 1 - a$$

which completes the proof. $\square$

57.3 IV, 3.6, 3.13, 5.4, Extra Problems

57.3.1 Exercise IV.3.6: Curves of Degree 4

(a) If X is a curve of degree 4 in some $\mathbf{P}^n$, show that either

1. *$g = 0$, in which case X is either the rational normal quartic in $\mathbf{P}^4$ (Ex. 3.4) or the rational quartic curve in $\mathbf{P}^3$ (II, 7.8.6), or*
2. *$X \subset \mathbf{P}^2$, in which case $g = 3$, or*
3. *$X \subset \mathbf{P}^3$ and $g = 1$.*

(b) In the case $g = 1$, show that X is a complete intersection of two irreducible quadric surfaces in $\mathbf{P}^3$ (I, Ex. 5.11).

Proof. (a) First suppose $n \geq 4$. If X is not contained in any $\mathbf{P}^{n-1}$ then since $4 \leq n$ (Ex. 3.4b) implies $n = 4$, $g(X) = 0$, and X differs from the rational normal curve of degree 4 only by an automorphism of $\mathbf{P}^4$. (I take this to mean that X *is* the rational normal curve of degree 4.) Thus when $n \geq 4$ we have proved that case (1) occurs.

Next suppose $n = 3$. Then X is a degree 4 curve in $\mathbf{P}^3$. If X is contained in some $\mathbf{P}^2$ then $g = \frac{1}{2}(d-1)(d-2) = 3$ and so case (2) occurs. If X is not contained in any $\mathbf{P}^2$ then (Ex. 3.5 b) implies

$$g < \frac{1}{2}(d-1)(d-2) = 3.$$

Thus g is either 0, 1, or 2. If $g = 0$ then X is the rational quartic curve in $\mathbf{P}^3$ which is case (1). If $g = 1$ then X falls into case (3). If $g = 2$ then $\mathcal{O}_X(1)$ is a very ample divisor of degree 4 (3.3.2) on a genus 2 curve contrary to (Ex. 3.1) which asserts that the degree of a very ample divisor on a curve of genus 2 is at least 5.

Next suppose $n = 2$. Then X is a degree 4 curve in $\mathbf{P}^2$ so X has genus 3 and falls into case (2).

The case $n = 1$ cannot occur since $\mathbf{P}^1$ contains no curve of degree 4.

(b) A curve C of genus 1 has g^1_2's since Riemann-Roch guarantees that the complete linear system associated to any divisor of degree 2 is a g^1_2. There are infinitely many divisors of degree 2 which is not linearly equivalent. This is because $P + Q \sim P + R$ iff $Q \sim R$. Since C is not rational there are infinitely many points Q and R with $Q \not\sim R$.

Given two distinct g^1_2's take the product of the corresponding morphisms to obtain a map $\varphi : C \to \mathbf{Q}$ where Q is the quadric surface in $\mathbf{P}^3$. The diagram is

$$\begin{array}{ccc} C & \longrightarrow & \mathbf{P}^1 \\ \downarrow & \searrow \varphi & \uparrow \\ \mathbf{P}^1 & \longleftarrow & C_0 \subset Q \end{array}$$

Let C_0 denote the image of C under φ. Note that C_0 is not a point so the type (a,b) of C_0 is defined. Let e be the degree of φ. Then by analyzing how certain divisors pull back one sees that $ae = 2$ and $be = 2$. The only possibilities are that C_0 is of type $(1,1)$ or of type $(2,2)$. If C_0 is of type $(1,1)$ then C_0 must be nonsingular because of the relation between the arithmetic genus of C_0 and that of its normalization. So in this case $C_0 \cong \mathbf{P}^1$ and the projections $Q \to \mathbf{P}^1$ are injective when restricted to C_0. This implies that the two maps coming from the distinct g^1_2's collapse the same points, a contradiction.

Thus C_0 is of type $(2,2)$ and $e=1$. Because C has genus 1 and C_0 has arithmetic genus $(2-1)(2-1)=1$ it follows that $C \cong C_0$. We now know that C embeds as a type $(2,2)$ curve on Q. Since a curve of type (a,a) on Q is a complete intersection we are done.

Another way to do this problem is to somehow embed X into $\mathbf{P}^3$ as a degree 4 curve then consider the exact sequence

$$0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbf{P}^3}(2) \to \mathcal{O}_X(2) \to 0.$$

Taking cohomology yields the exact sequence of vector spaces

$$0 \to H^0(\mathcal{I}_X(2)) \to H^0(\mathcal{O}_{\mathbf{P}^3}(2)) \to H^0(\mathcal{O}_X(2)) \to \cdots.$$

Since X has degree 4 it follows that $\dim H^0(\mathcal{O}_X(2)) = 8$. Furthermore $\dim H^0(\mathcal{O}_{\mathbf{P}^3}(2)) = 10$ so we see that

$$\dim H^0(\mathcal{I}_X(2)) \geq 2.$$

This means that there exist linearly independent homogeneous polynomials f and g of degree 2 such that $X \subset Z(f)$ and $X \subset Z(g)$. Since X has degree 4 and $Z(f) \cap Z(g)$ has degree 4 the fact that $X \subset Z(f) \cap Z(g)$ implies that $X = Z(f) \cap Z(g)$. This is seen by looking at the appropriate Hilbert polynomials. □

57.3.2 Exercise IV.3.12

For each value of $d = 2,3,4,5$ and r satisfying $0 \leq r \leq \frac{1}{2}(d-1)(d-2)$, show that there exists an irreducible plane curve of degree d with r nodes and no other singularities.

Proof. I did the first few by finding explicit equations. It might have been better to do everything by abstract general methods but it was a good exercise to search for defining equations.

$d=2, r=0$: Take $f = x^2 - yz = 0$. This works in any characteristic since the partials are: $f_x = 2x, f_y = -z, f_z = -y$ so if $f_y = f_z = 0$ then $z = y = 0$ so $x = 0$. But $(0,0,0)$ is not a point.

$d=3, r=0$: When $\operatorname{char} k \neq 3$ take $x^3+y^3+z^3 = 0$ which is clearly nonsingular. For $\operatorname{char} k = 3$ take $x^2y + z^2y + z^3 + y^3 = 0$. This is nonsingular as was proved in my solution to (I, Ex. 5.5).

$d=3, r=1$: Let $f = xyz + x^3 + y^3$, then $f_x = yz + 3x^2$, $f_y = xz + 3y^2$ and $f_z = xy$. Thus a singular point must have $x=0$ or $y=0$. If $x=0$ then from $f=0$ we see that $y=0$. Thus $(0:0:1)$ is the only singularity and it is clearly nodal. Note that this curve works in characteristic 3 as well.

$d=4, r=0$: When $\operatorname{char} k \neq 2$ take $x^4+y^4+z^4=0$. If $\operatorname{char} k = 2$ take $x^3y+z^3y+z^4+y^4 = 0$ as in (I, Ex. 5.5).

$d=4, r=1$: If $\operatorname{char} k \neq 2$ take $f = xyz^2 + x^4 + y^4 = 0$. Then $f_x = yz^2 + 4x^3$, $f_y = xz^2 + 4y^3$, and $f_z = 2xyz$. The only singular point is $(0:0:1)$ which is a node. When $\operatorname{char} k = 2$ take $f = xyz^2 + x^3z + y^4 = 0$. The partials are $f_x = yz^2 + x^2z$, $f_y = xz^2$, and

$f_z = x^3$. Thus a singular point must satisfy $x = 0$. Then $f = 0$ implies $y = 0$. Thus the only singular point is $(0 : 0 : 1)$ which is a node.

$d = 4, r = 2$: Let C be your favorite genus 1 curve. Let D be a divisor of degree 4, then by (3.3.3) D is very ample. By Riemann-Roch,

$$\dim |D| - \dim |K - D| = 4 + 1 - 1$$

so $|D|$ gives rise to embedding of C as a degree 4 curve in $\mathbf{P}^3$. Using (3.11) project C onto the plane to obtain a plane curve of degree 4 with only nodal singularities. Since the genus of the normalization is 1 it follows that there are exactly 2 nodes.

$d = 4, r = 3$: Embed $\mathbf{P}^1$ as the degree 4 rational normal curve in $\mathbf{P}^4$. Then use (3.5) and (3.10) to project into $\mathbf{P}^2$ to get a curve X of degree 4 in $\mathbf{P}^2$ having only nodes for singularities. Since

$$0 = g(\tilde{X}) = \frac{1}{2}(4-1)(3-1) - \text{ number of nodes}$$

it follows that X has 3 nodes.

$d = 5, r = 0$: When $\operatorname{char} k \neq 5$ take $x^5+y^5+z^5 = 0$. If $\operatorname{char} k = 5$ take $x^4y+z^4y+z^5+y^5 = 0$.

$d = 5, r = 1$: When $\operatorname{char} k \neq 5$ take $f = xyz^3 + x^5 + y^5 = 0$. Then $f_x = yz^3 + 5x^4$, $f_y = xz^3 + 5y^4$, and $f_z = 3xyz^2$. The only singular point is $(0 : 0 : 1)$ which is a node. In characteristic 5 let $f = xyz^3 + x^5 + y^5 + x^3y^2$. Then $f_x = yz^3 + 3x^2y^2$, $f_y = xz^2 + 2x^3y$, and $f_z = 3xyz^2$ so the only singular point is $(0 : 0 : 1)$ which is clearly a nodal singularity.

$d = 5, r = 2$: By (Ex 5.4 a) a genus 4 curve which has two distinct g^1_3's gives rise to a plane quintic with two nodes. The curve of type $(3, 3)$ on the quadric surface is such a curve.

$d = 5, r = 3$. I have not found one yet. If I could find a degree 5 space curve of genus 3 I would win. But by Theorem 6.4 no such space curve exists.

$d = 5, r = 4$: Let C be a curve of genus 2. By Halphen's theorem there exists a nonspecial very ample divisor D of degree 5. Then

$$\dim |D| = 5 + 1 - 1 - 1 = 3$$

so C embeds into $\mathbf{P}^3$ as a curve of degree 5. Project to $\mathbf{P}^2$ to obtain a curve X of degree 5 whose singularities are all nodal and whose normalization has genus 2. It follows that X has 4 nodes.

$d = 5, r = 5$: Pick your favorite curve of genus 1. By (3.3.3) there exists a very ample nonspecial divisor of degree 5. As usual, project to obtain a degree 5 plane curve with singularities only nodes and normalization of genus 1. It follows that there must be exactly 5 nodes.

$d = 5, r = 6$: Embed $\mathbf{P}^1$ in $\mathbf{P}^5$ as a curve of degree 5 (Ex. 3.4), and then project it into $\mathbf{P}^2$ to get a curve X of degree 5 in $\mathbf{P}^2$ having only nodes as singularities Since

$$0 = g(\tilde{X}) = \frac{1}{2}(5-1)(4-1) - \text{ number of nodes}$$

the number of nodes must be 6. □

We can also obtain the following general result.

Proposition 57.7. *If r and d are such that*

$$\frac{1}{2}(d-1)(d-2) + 3 - d \leq r \leq \frac{1}{2}(d-1)(d-2)$$

then there exists a plane curve of degree d which has exactly r singularities all of which are nodes.

Proof. By (6.2) if $d \geq g+3$ then there exists a curve in $\mathbf{P}^3$ of genus g and degree d. Using (3.11) project this curve onto the plane to obtain a curve of degree d with

$$r = \frac{1}{2}(d-1)(d-2) - g$$

singularities all of which are nodes. We can carry out this process so long as $r \leq \frac{1}{2}(d-1)(d-2)$ and $d \geq g+3$, that is, as long as

$$r = \frac{1}{2}(d-1)(d-2) - g \geq \frac{1}{2}(d-1)(d-2) + 3 - d.$$

□

57.3.3 Exercise IV.5.4

Another way of distinguishing curves of genus g is to ask, what is the least degree of a birational plane model with only nodes as singularities (3.11)? Let X be nonhyperelliptic of genus 4. Then:

(a) if X has two g^1_3's, it can be represented as a plane quintic with two nodes, and conversely;

(b) if X has one g^1_3, then it can be represented as a plane quintic with a tacnode (I, Ex. 5.14d), but the least degree of a plane representation with only nodes is 6.

Proof. (a) Summary: If X is nonhyperelliptic with two g^1_3's then X is type $(3,3)$ in the quadric so projecting through a point on X gives a plane quintic model with exactly two nodal singularities.

Suppose X is nonhyperelliptic and X has two g^1_3's. Let p and p' be the degree 3 maps $X \to \mathbf{P}^1$ determined by the two g^1_3's. Let

$$\varphi : X \to \mathbf{P}^1 \times \mathbf{P}^1 = Q \subset \mathbf{P}^3$$

be their product. Let $X_0 = \varphi(X)$ be the image of X, and let (a,b) be the type of X_0. Letting e be the degree of φ we see that $ea = 3$ and $eb = 3$. This implies that either $a = b = 1$ and $e = 3$ or $a = b = 3$ and $e = 1$.

First suppose $a = b = 1$ and $e = 3$. Then X_0 is nonsingular of genus 0 and the two projection maps $X_0 \to \mathbf{P}^1$ are injective. This implies p and p' collapse the same points, a contradiction.

Thus $a = b = 3$ and $e = 1$. Because the arithmetic genus of X_0 is $(a-1)(b-1) = 4$ and X has genus 4 we see that X_0 must be nonsingular. Thus X embeds as a type $(3,3)$ curve on the quadric surface. Henceforth view X as embedded in Q.

Pick a point P_0 on $X \subset Q$ and fix a copy of $\mathbf{P}^2$. Map X to $\mathbf{P}^2$ by projection through P_0. If two points P, Q project to the same point then P_0, P, Q are collinear. I claim that this implies the line L through P_0, P, Q is contained in Q. To see this let H by a hyperplane containing L. Then either $H.Q$ is two copies of a line or a degree 2 curve. If $H.Q$ is a degree 2 curve then $L.(H.Q)$ consists of at most 2 points, a contradiction since the points P_0, P, Q are all contained in $L.(H.Q)$. Thus $H.Q$ is two copies of a line. Since P_0, P, Q are contained in $H.Q$ we see that $H.Q = 2L$ and hence that L is contained in Q. Since there are exactly 2 lines on Q through any given point of Q the image of X under projection can have at most 2 singularities.

To show that the singularities are nodes we show that three or more points can not collapse under projection. [[This is not quite enough. I do not know how to show that there is not some other unusual singularity.]] Suppose some three points are collapsed under projection. Then there exists points P, Q, R on X such that P_0, P, Q, R are all collinear. Now X is of type $(3, 3)$ so X is a complete intersection $Q.F_3$ where F_3 is some degree 3 surface. Since F_3 has degree 3 and contains P_0, P, Q, R it must contain the line through them. (Take a plane containing the line, and apply an argument as above.) Similarly Q contains the line through P_0, P, Q, R so X contains this line, a contradiction.

Since the genus of the normalization of the image X_0 of X in $\mathbf{P}^2$ is 4 and X_0 has exactly two nodes as singularities it follows that X_0 has degree 5.

The converse. Suppose we are given a plane quintic curve C with two nodes and no other singularities. Then the normalization has genus

$$g(\tilde{C}) = \frac{1}{2}(5-1)(4-1) - 2 = 4.$$

Thus C represents a curve X of genus 4. Let P_0 be one of the two nodes, then since C has degree 5 a line through P_0 intersect C in 3 other points. This gives a degree 3 map from C to $\mathbf{P}^1$. Since this map is defined on a nonempty open subset of X it extends to a degree 3 morphism of X into $\mathbf{P}^3$. This gives a g_3^1 on X. The other node gives a different g_3^1 so X has two distinct g_3^1's.

We must also show that X is not hyperelliptic. Suppose X is hyperelliptic so X has a g_2^1. Let p be the map to $\mathbf{P}^1$ corresponding to this g_2^1 and let p' be the map to $\mathbf{P}^1$ corresponding to some g_3^1. Let $\varphi = p \times p' : X \to Q \subset \mathbf{P}^3$ be their product. Let $X_0 = \varphi(X)$ be the image of X and suppose X_0 has type (a, b). Let e be the degree of φ. Then $ea = 2$ and $eb = 3$. Thus $e = 1$ so X is birational to the normalization of X_0. But X_0 has arithmetic genus $(2-1)(3-1) = 2 < 4$ which is a contradiction.

(b) Suppose C is a nonhyperelliptic curve with exactly one g_3^1. By looking at twists of certain exact sequences we showed in class that C lies on the (singular) quadric cone. We showed furthermore that C is a complete intersection $Q_{\text{cone}}.F_3$ of Q_{cone} with some cubic hypersurface F_3. Pick a point P_0 on C and a copy of $\mathbf{P}^2 \subset \mathbf{P}^3$. Projection through P_0 defines a map $C \to \mathbf{P}^2$. If two points P, Q collapse then the three points P_0, P, Q are collinear. Let L be the line determined by P_0, P, Q. Since Q_{cone} has degree 2 and L intersects Q_{cone} in at least three points it follows that L lies on Q_{cone}. Since there is only one line through P_0 which lies on Q_{cone} it follows that projection defines a birational map of C to plane curve C_0 which has exactly one singularity.

If 3 points P, Q, R are collapsed by projection then the four points P_0, P, Q, R are collinear. Let L be the line through them. Then L must lie in F_3 and L must lie in Q which is a contradiction. This shows that the singular point on C_0 is a double point.

I do not know how to show it must be a tacnode.

The Converse:

If there is a quintic plane representation of degree less than 6 with only nodes then there must be at least two nodes because of the formula for the genus of the normalization. But each node gives rise to a distinct g_3^1. Since there is a unique g_3^1 on X this is a contradiction. □

57.3.4 Extra Problem 3, by William Stein

Suppose C is hyperelliptic and $g \geq 3$. Then there does not exist a g_3^1 on C.

Proof. Suppose that C has a g_3^1 and let $p : C \to \mathbf{P}^1$ be the corresponding morphism. Let $p' : C \to \mathbf{P}^1$ be the morphism corresponding to a g_2^1 on the hyperelliptic curve C. Let

$$\varphi = p \times p' : C \to \mathbf{P}^1 \times \mathbf{P}^1 = Q \subset \mathbf{P}^3.$$

Let (a, b) be the type of $C_0 = \varphi(C)$. If e denotes the degree of φ then $ea = 2$ and $eb = 3$. We see this by looking at how divisors of types $(1, 0)$ and $(0, 1)$ pull back. Thus $e = 1$, $a = 2$ and $b = 3$. Now C is isomorphic to the normalization of C_0 which has arithmetic genus $(2 - 1)(3 - 1) = 2 < 3$ so C has genus less than 3, a contradiction. $\square$

57.3.5 Extra Problem 4, by Nghi Nguyen

If C is a non-hyperelliptic curve of genus $g \geq 4$, show that C has at most a finite number of g^1_3's.

Proof. This proof is the work of Nghi although the write up is my own. (Therefore any mistakes are my responsibility.)

Summary. For a fixed point P_0 there are only finitely many g^1_3's arising from a divisor D of the form $D = P_0 + Q + R$. If some g^1_3 is defined by a divisor D, then $D - P_0$ is linearly equivalent to an effective divisor $Q + R$ so $D \sim P_0 + Q + R$. Combining this with the first assertion implies that there are only finitely many g^1_3's.

Step 1. Fix a point P_0, then we show that there are only finitely many g^1_3's arising from a divisor D of the form $P_0 + Q + R$. Suppose $D = P_0 + Q + R$ is a divisor such that $|D|$ is a g^1_3.

Since C is non-hyperelliptic the canonical divisor K is very ample. Thus $K - P_0$ is base point free. Let $\varphi : X \to \mathbf{P}^{g-2}$ be the morphism determined by $K - P_0$. By Riemann-Roch

$$\dim |P_0 + Q + R| - \dim |K - P_0 - Q - R| = 3 + 1 - g$$

so since $\dim |P + Q + R| = 1$,

$$\dim |K - P_0 - Q - R| = g - 3.$$

But K is very ample so $\dim |K - P_0 - Q| = \dim |K| - 2 = g - 3$. Thus

$$\dim |K - P_0 - Q - R| = \dim |K - P_0 - Q|$$

so R is a basepoint of $K - P_0 - Q$. This means that $\varphi(R) = \varphi(Q)$.

Let $X_0 = \varphi(X) \subset \mathbf{P}^{g-2}$. Let $\mu = \deg \varphi$ and let $d = \deg X_0$. Then

$$\mu d = \deg(K - P_0) = 2g - 3.$$

If $\mu > 1$ then $d \leq \frac{2g-3}{3}$ since μ is not 2 because $2g - 3$ is odd. But $\frac{2g-3}{3} < g - 2$ since $g \geq 4$. Thus $d < g - 2$.

Let $\tilde{X}_0$ be the normalization of the degree d curve $X_0 \subset \mathbf{P}^{g-2}$. Let h be the linear system of degree d corresponding to the morphism

$$\tilde{X}_0 \to X_0 \subset \mathbf{P}^{g-2}.$$

Then h is a linear system of degree d and dimension $g - 2$. Thus by (Ex. 3.4) we conclude that $g - 2 \leq d$. This contradicts the conclusion $d < g - 2$ which followed from our assumption that $\mu > 1$. Thus $\mu = 1$.

Since $\mu = 1$ the map φ is birational so it can collapse only finitely many points. This means that there are only finitely many choices for $D = P_0 + Q + R$ so that $|D|$ is a g^1_3.

Step 2. Suppose $|D|$ is a g^1_3. Then $\dim |D| = 1$ and since $|D|$ is basepoint free

$$\dim |D - P_0| = \dim |D| - 1 = 0.$$

Thus there exists an effective divisor $Q + R$ such that

$$D - P_0 \sim Q + R,$$

so $D \sim P_0 + Q + R$. This shows that every g_3^1 is defined by an effective divisor which contains P_0. By step 1 we know there are only finitely many such g_3^1 so we conclude that C has only finitely many g_3^1's. $\square$