Algebraic Geometry Homework

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1 Solutions

Problem 1 (II.2.14(a)) *Let S be a graded ring. Show that $Proj\, S = \emptyset$ iff every element of S_+ is nilpotent.*

PROOF. This is equivalent to showing that the nilradical of S is equal to the intersection of all homogenous primes of S. Indeed, if every element of S_+ is nilpotent then every homogenous prime contains S_+ so $Proj\, S = \emptyset$. Conversely, if $Proj\, S = \emptyset$, then every homogenous prime contains S_+ so the nilradical of S contains S_+ so that every element of S_+ is nilpotent.

It remains to show that the intersection of all homogenous primes of S is the nilradical of S. Using Zorn's lemma we can show that if I is a proper homogenous ideal then there is at least one maximal homogenous ideal containing I. (The proof preceeds just as on page 2 of Matsumura except one notes that the union of a chain of *homogenous* ideals is homogeneous. If $I_1 \subset I_2 \subset \cdots$ is a chain of homogeneous ideals, then $\cup_{n=1}^{\infty} I_n$ is an ideal and if $x \in \cup_{n=1}^{\infty} I_n$ then $x \in I_n$ for some n, so the homogenous components of x are in I_n, so they are in $\cup_{n=1}^{\infty} I_n$, so $\cup_{n=1}^{\infty} I_n$ is homogeneous.)

Theorem 1 *Let T be a multiplicative set and I a homogeneous ideal disjoint from T; then there exists a homogeneous prime ideal containing I and disjoint from T.*

PROOF. Using Zorn's lemma we see that the set of homogeneous ideals disjoint from T and containing I contains a maximal element, say P. Then P is prime. For if $x \notin P$, $y \notin P$ are homogenous, then $P + (x)$ and $P + (y)$

are both homogenous and so they meet T, so their product also meets T. However,

$$(P+(x))(P+(y)) \subset P+(xy)$$

so $xy \notin P$ since P does not meet T.

Theorem 2 *The nilradical of S is the intersection of the homogeneous primes of S.*

PROOF. Suppose x is in the nilradical of S so that x is nilpotent, say $x^n = 0$. If I is a homogenous prime then $x^n = 0 \in I$ so, by induction, $x \in I$. Conversely, suppose x is not nilpotent. Then $T = \{1, x, x^2, \ldots\}$ is a multiplicative set disjoint from (0). So, by the above theorem, there is a homogeneous prime ideal containing (0) disjoint from T. Thus x is not in the intersection of all homogeneous prime ideals of S.

Problem 2 (II.2.14(b)) *Let $\varphi : S \to T$ be a graded homomorphism of graded rings. Let $U = \{p \in Proj\, T : p \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of $Proj\, T$, and show that φ determines a natural morphism $f : U \to Proj\, S$.*

PROOF. U is open because $Proj\, T - U = \{p \in Proj\, T : p \supseteq \varphi(S_+)\} = \{p \in Proj\, T : p \supseteq T\varphi(S_+)\} = V(T\varphi(S_+))$ and the ideal $T\varphi(S_+)$ is homogenous because it is generated by the homogeneous elements $\{\varphi(f) : f \in S_+ \text{ is homogeneous}\}$.

The natural morphism $f : U - Proj\, S$ is defined as follows. As a map on topological spaces we specify that, for $x \in U$, $f(x) = \varphi^{-1}(x)$. Because of the way U was chosen and since φ is homogeneous f maps U into $Proj\, S$, so f is well-defined. If $V(a)$ is a closed subset of $Proj\, S$ with a homogeneous, then $f^{-1}(V(a)) = \{p \in U : \varphi^{-1}(p) \supseteq a\} = \{p \in U : p \supseteq \varphi(a)\} = V(T\varphi(a))$. As above, $T\varphi(a)$ is homogeneous so $f^{-1}(V(a))$ is closed so f is continuous.

To define the associated map $f^\# : \mathcal{O}_{Proj\, S} \to f_*\mathcal{O}_U$ of sheaves let $V \subset Proj\, S$ be open. Then an element of $\mathcal{O}_{Proj\, S}(V)$ is a map $s : V \to \bigsqcup_{p \in V} S_{(p)}$ such that s is locally a quotient of elements of S. We specify that $f^\#(s) = \varphi \circ s \circ f : f^{-1}(V) \to \bigsqcup_{p \in f^{-1}(V)} T_{(p)}$. To see that $f^\#(s) \in f_*\mathcal{O}(V)$ we must check that it is locally a quotient. So suppose $p \in f^{-1}(V)$. Let $W \subset V$ be an open neighborhood of $f(p)$ on which s is represented as a quotient. Then $f^\#(s)$ is represented as a quotient on the open set $f^{-1}(W)$, as required. Since $f^\#$ respects the restriction maps we see that f is a morphism.

Problem 3 (II.2.14(c)) *f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$. Show that $U = ProjT$ and the morphism $f : ProjT \to ProjS$ is an isomorphism.*

PROOF. To see that $U = ProjT$ note that if $p \in ProjT$ but $p \notin U$ then $p \supseteq \varphi(S_+)$. In particular, $p \supseteq \bigoplus_{d \geq d_0} T_d$, so $p \supseteq (T_+)^d$ so, since p is prime, $p \supseteq T_+$, a contradiction.

Let $\{g_\alpha\}$ be a set of generators of T_+. Then $\cup_\alpha D_T(g_\alpha) = \cup_\alpha \{x \in ProjT : g_\alpha \notin x\} = ProjT$ since every prime in $ProjT$ must omit some g_α. Since $g_\alpha \notin x$ iff $g_\alpha^{d_0} \notin x$ for x prime, we may replace the g_α by elements of $T_{\geq d_0}$ and still have a cover of $ProjT$ by distinguished open sets.

Our strategy is as follows. We first show that $f|_{D_T(g_\alpha)} : D_T(g_\alpha) \to D_S(\varphi^{-1}(g_\alpha)$ is an isomorphism for each α and then show that the open sets $D_S(\varphi^{-1}(g_\alpha))$ cover $ProjS$. Then showing that f is injective completes the proof.

Let $g = g_\alpha$ be one of our g_α. By Proposition 2.5, $D_T(g) \cong SpecT_{(g)}$ and so $f' = f|_{D_T(g)}$ is a morphism of affine schemes $f' : SpecT_g \to SpecS_{(\varphi^{-1}(g))}$. This map is induced by $\overline{\varphi} : S_{(\varphi^{-1}(g))} \to T_{(g)}$ where $\overline{\varphi}$ is the localization of the ring homomorphism $\varphi : S \to T$. So we just need to verify that $\overline{\varphi}$ is an isomorphism. Suppose $\overline{\varphi}(a/b) = 0$. Then $\overline{\varphi}(a\varphi^{-1}(g)/b\varphi^{-1}(g)) = \overline{\varphi}(a/b) = 0$ so $\varphi(a\varphi^{-1}(g))/\varphi(b\varphi^{-1}(g)) = 0$ in $T_{(g)}$ so there is n such that $g^n\varphi(a\varphi^{-1}(g)) = 0$ in T so $\varphi(a\varphi^{-1}(g^{n+1}) = 0$. Thus $a\varphi^{-1}(g)^{n+1} = 0$ since φ is an isomorphism in high enough degree. Thus $a = 0$ in $S_{(\varphi^{-1}(g))}$, so $a/b = 0$ in $S_{(\varphi^{-1}(g))}$. This shows that $\overline{\varphi}$ is injective. To see that $\overline{\varphi}$ is surjective let $a/g^n \in T_{(g)}$. Then $\varphi^{-1}(ag)/\varphi^{-1}(g^{n+1})$ is a well-defined element of $S_{(\varphi^{-1}(g))}$ and $\overline{\varphi}(\varphi^{-1}(ag)/\varphi^{-1}(g^{n+1})) = ag/g^{n+1} = a/g^n$, which shows that $\overline{\varphi}$ is surjective.

Next we verify that $\cup_{g_\alpha} D_S(\varphi^{-1}(g_\alpha) = ProjS$. Suppose $x \notin D_S(\varphi^{-1}(g_\alpha)$ for all α. Then $\varphi^{-1}(g_\alpha) \in x$ for each α, so, since we may assume that the g_α generate $T_{\geq d_0}$, and x is prime, $\varphi^{-1}(T_{\geq d_0}) \subseteq x$, a contradiction since $S_+ \not\subseteq x$.

Next we show that the induced map $f : ProjT \to ProjS$ is injective. Let $p, q \in ProjT$ and suppose $f(p) = f(q)$. Then $\varphi^{-1}(p) = \varphi^{-1}(q)$ so, since φ is an isomorphism, for $d \geq d_0$ we see that $p \cap T_d = q \cap T_d$. So if $a \in p$ is homogeneous then $a^n \in p \cap T_d$ for some n and $d \geq d_0$. So $a^n \in q \cap T_d$ so $a^n \in q$ so $a \in q$. Likewise $a \in q$ implies $a \in p$. Thus $p = q$ so f is injective.

Finally, no solution is complete without an actual example of a map $\varphi : S \to T$ which satisfies the hypothesis of the theorem. Let $T = k[x, y]$ and let $S = T_0 + T_2 + \cdots$ and let $\varphi : S \hookrightarrow T$ be the inclusion map. Then φ is graded and an isomorphism for $d \geq 2$. But φ is not an isomorphism.

Problem 4 (II.2.14(d)) *Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \cong Proj\, S$.*

PROOF. Define $f : t(V) \to Proj\, S$ by f takes a point $x \in t(V)$ to its homogeneous ideal $I(x) \in Proj\, S$. By exercise I.2.4 f is injective and surjective hence a bijection. Furthermore, $x \supseteq y$ iff $f(x) \subseteq f(y)$ so f and f^{-1} send closed sets to closed sets hence f is a homeomorphism.

To define $f^{\#}$ let U be an open subset of $Proj\, S$. Then we must define $f^{\#}$ so that (notation as in the proof of proposition 2.6) $f^{\#} : \mathcal{O}_{Proj\, S}(U) \to f_* \mathcal{O}_{t(V)}(U) = \mathcal{O}_{t(V)}(f^{-1}(U)) = \mathcal{O}_V(\alpha^{-1}(f^{-1}(U)))$. Let $s \in Proj_S(U)$. Then s can locally be represented in the form g/h where $g, h \in S$ have the same degree and h is nonzero on the appropriate subset of V. Thus s naturally defines an element of $\mathcal{O}_V(\alpha^{-1}(f^{-1}(U)))$ and any element of $\mathcal{O}_V(\alpha^{-1}(f^{-1}(U)))$ defines an element of $\mathcal{O}_{Proj\, S}(U)$. Thus $f^{\#}$ is an isomorphism. [I'm glossing over a lot of details!]

Problem 5 (II.2.16(a)) *Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal m_x of the local ring $\mathcal{O}_x$. (a) If $U = Spec B$ and $\overline{f} \in B = \Gamma(U, \mathcal{O}_{X|U})$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$ so X_f is an open subset of X.*

PROOF. Note that $D(\overline{f}) = \{x \in U : \overline{f} \notin x\} = \{x \in U : \overline{f}_x \notin m_x\} = \{x \in U : f_x \notin m_x\} = U \cap X_f$. Thus $U \cap X_f$ is an open subset of U. Now let $\{U_\alpha\}$ be an affine open over of X. Then $X_f = \cup_\alpha (U_\alpha \cap X_f)$ is the union of open sets, hence open.

Problem 6 (II.2.16(b)) *Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.*

PROOF. Using the fact that X is quasi-compact we can find a finite cover $\{U_i = Spec B_i\}_{i=1}^m$ of X by affine open sets. Since $a|_{X_f}$ is 0, the image of a in $\mathcal{O}_{X_f \cap U_i} = (B_i)_{\overline{f}}$ is 0. Thus there exists n_i such that $\overline{f}^{n_i} \overline{a} = 0$ in B_i. That is, $f^{n_i} a|_{X_f \cap U_i}$ is 0. Letting $n = \max\{n_1, \ldots, n_m\}$ we see that $f^n a|_{X_f \cap U_i}$ is 0 for each i whence $f^n a = 0$ in $A = \Gamma(X, \mathcal{O}_X)$.

Problem 7 (II.2.16(c)) *Assume X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.*

PROOF. Write $U_i = Spec B_i$. Then $b|_{U_i} \in (B_i)_f$ so, from the definition of $(B_i)_f$, there exists n_i such that $f^{n_i} b|_{U_i} \in B_i = \mathcal{O}_X(U_i)$. Let $N = \max\{n_i\}$ and let $g_i = f^N b|_{U_i} \in \mathcal{O}_X(U_i)$. Then $g_i|_{X_f \cap U_i \cap U_j} = f^N b|_{U_i \cap U_j \cap X_f} = g_j|_{X_f \cap U_i \cap U_j}$ so $(g_i - g_j)|_{(U_i \cap U_j)_f} = 0$. By part (b) since $U_i \cap U_j$ is quasi-compact there is an integer n_{ij} such that $f^{n_{ij}}(g_i - g_j) = 0$ in $\mathcal{O}_{U_i \cap U_j}$. Let $M = \max\{n_{ij}\}$, and let $h_i = f^M g_i$. Then $h_i \in \mathcal{O}(U_i)$ for each i and $h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j}$ so we can find $h \in \Gamma(X, \mathcal{O}_X)$ such that $h|_{U_i} = h_i$. But, for each i, $h|_{X_f \cap U_i} = f^M g_i|_{X_f \cap U_i} = f^M f^N b|_{X_f \cap U_i} = f^{M+N} b|_{X_f \cap U_i}$ so by uniqueness (sheaf axiom iii), $h|_{X_f} = f^{M+N} b$, as desired.

Problem 8 (II.2.16(d)) *With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.*

PROOF. Define a homomorphism $\varphi : A_f \to \Gamma(X_f, \mathcal{O}_{X_f})$ by $\varphi(a/f^n) = a|_{X_f}/f^n|_{X_f}$. This is well-defined because the stalk f_x is invertible in each local ring $\mathcal{O}_x$ for each $x \in X_f$. φ is a homomorphism since restriction is a homomorphism. Suppose $\varphi(a/f^n) = 0$, then $a|_{X_f}/f|_{X_f}^n = 0$ so $a|_{X_f} = 0$. By part (b) there exists m such that $f^m a = 0$, so a is 0 in A_f, whence φ is injective. Suppose $b \in \Gamma(X_f, \mathcal{O}_{X_f})$, then by part (c) there exists n such that $f^n b$ is the restriction of some $a \in \Gamma(X, \mathcal{O}_X)$ to X_f. Then $\varphi(a/f^n) = a|_{X_f}/f|_{X_f}^n = f^n b/f^n|_{X_f} = b$ so φ is surjective, which establishes the desired isomorphism.

Problem 9 (II.2.17(a)) *A Criterion for Affineness. Let $f : X \to Y$ be a morphism of schemes, and assume that Y can be covered by open sets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.*

PROOF. Define a morphism $g : Y \to X$ as follows. On each open set U_i let g_i be the morphism inverse to $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$. Then $g_i|_{U_i \cap U_j} = f|_{f^{-1}(U_i) \cap f^{-1}(U_j)} = g_j|_{U_i \cap U_j}$ so, as in step 3 of the proof of Theorem 3.3, we can glue the morphisms g_i to obtain a morphism $g : Y \to X$ such that $g|_{U_i} = g_i$. As a map of spaces g is clearly inverse to f. Since each g_i is an isomorphism, the induced maps $g^\#$ on stalks are isomorphisms so the induced map on sheaves is an isomorphism. Thus g is an isomorphism.