Algebraic Geometry Homework
II.3 #'s 6,7,8,12,14
II.4 #'s 2, 4

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Problem 1 (3.6) *Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. Call it $K(X)$. Show also that if $U = \text{Spec } A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.*

Proof. Let $U = \text{Spec } A$ be any nonempty open affine subset of X. Then since the closure of a generic point of X is all of X, every open set must contain a generic point. Thus if ξ is a generic point, then $\xi \in U$. But A is an integral domain so (0) is the unique generic point of U, whence $\xi = (0)$. This shows the generic point is unique if it exists. Since X is integral it is irreducible so every open set intersects U. Thus every open set contains $(0) \in \text{Spec } A = U$, so X actually contains a generic point $\xi = (0)$. Furthermore, $\mathcal{O}_\xi \cong A_{(0)}$ is the quotient field of A.

Problem 2 (3.7) *Let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.*

Proof. Let $U = \text{Spec } B$ be an open affine subset of Y which contains the generic point of Y. Let $V = \text{Spec } A$ be an open affine subset of $f^{-1}(U)$. Since f is of finite type A is a finitely generated B-algebra. The generic point of X is in V since every open set contains the generic point. Let $\varphi : B \to A$ be the homomorphism corresponding to the induced morphism of affine schemes $f : V \to U$. Since f is dominant, we know that φ is injective. The induced map on stalks then gives an inclusion of function fields $K(Y) = B_{(0)} \hookrightarrow A_{(0)} = K(X)$. Since A is a finitely generated B-algebra, $K(X)$ is a finitely generated field extension of $K(Y)$. If this field extension is not of finite degree then $K(X)$ must contain an element which is transcendental over $K(Y)$. Thus A must contain an element t which is transcendental over B. But then infinitely many primes of A lie over (0). Indeed, since $K(X)$ is finitely generated over $K(Y)$, $K(X)$ is a finite algebraic extension of $k(t)$ for some field k. Then (since the algebraic closure of a field is infinite), there are infinitely many irreducible polynomials in $k[t]$. Since $K(X)$ is finite algebraic over $k(t)$ infinitely many of these must remain irreducible in A. Multiplying through denominators this gives infinitely many irreducible elements of A which generate prime ideals which lie over (0). This would contradict the fact that f is generically finite. Thus $K(X)$ is finite over $K(Y)$.

Let $x_1, \ldots, x_n$ generate A as a B-algebra. Then, since $K(X)$ is finite over $K(Y)$ each x_i satisfies some polynomial f_i with coefficients in B. Let b be the product of all of the leading coefficients of the polynomials f_i. If b is a unit in B then so are all of the leading coefficients of the f_i so we can divide by them and hence assume the f_i are monic polynomials. If not, replace B by the localization B_b and repeat the whole argument with $U = \text{Spec } B_b$. In either case we may assume the f_i are monic from which we conclude that A is a finitely generated integral extension of B, thus A is a finite module over B.

Now let $U = \text{Spec } B$ be an open affine subset of Y which contains the generic point. Since f is of finite type we may write $f^{-1}(U) = \cup_{i=1,\ldots,n} V_i$ where each $V_i = \text{Spec } A_i$ is finitely generated B-algebra. By the work above we may shrink U so that we can assume each A_i is actually a finitely generated B-module. To complete the proof we need to show that there is a distinguised open subset of U (which necessarily contains the generic point) whose inverse image under f is an open affine which is the spectrum of a finitely generated B-module. Let $\varphi_i : B \to A_i$ be the homomorphism which induces $f|_{V_i}$. Since f is dominant, each φ_i is an injection. Thus we may, for notational convenience, identify B with it's images in the various A_i. The morphism f is then induced by the inclusion map $B \hookrightarrow A_i$.

Since $\cap_{i=1,\ldots,n} V_i$ is open we can, for each i, $1 \leq i \leq n-1$, find $\alpha_i \in A_i$ such that Spec $(A_i)_{\alpha_i} \subseteq \cap_{i=1,\ldots,n} V_i$. Since A_i is a finite module over B there is an integral equation

$$\alpha_i^n + b_{n-1}\alpha_i^{n-1} + \cdots + b_0 = 0$$

where each $b_j \in B$ and $b_0 \neq 0$. Let $b = \prod_{i=1,\ldots,n-1} b_i$. Then any prime of A_i which contains α_i must also contain b_i and hence b. Therefore Spec $(A_i)_b \subseteq$ Spec $(A_i)_\alpha$. We then have that $g^{-1}(\text{Spec } B_b) = \cup_{i=1,\ldots,n-1}\text{Spec } (A_i)_b \cup \text{Spec } (A_n)_b =$ Spec $(A_n)_b$. The latter equality follows since, for $1 \leq i \leq n-1$, Spec $(A_i)_b \subseteq \cap_{i=1,\ldots,n} V_i \cap f^{-1}(U_b) \subseteq f^{-1}(U_b) \cap V_n =$ Spec $(A_n)_b$. We thus see that U_b is a dense open subset of Y such that the morphism $f : f^{-1}(U_b) \to U_b$ is finite (since $f^{-1}(U_b)$ is the affine scheme Spec $(A_n)_b$ which a finite B_b-module). This completes the proof.

Problem 3 (3.8) *Normalization. Let X be an integral scheme. For each open affine subset $U =$ Spec A, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} =$ Spec $\tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the normalization of X. Show also that there is a morphism $\tilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\tilde{X}$. If X is of finite type over a field k, then the morphism $\tilde{X} \to X$ is a finite morphism.*

Proof.

We first verify the universal property for affine schemes where it is clear what the normalization is.

Proposition 1 *Suppose $X =$ Spec A, $\tilde{X} =$ Spec $\tilde{A}$ its normalization and $Z =$ Spec B is a normal integral scheme. Then every dominant morphism $f : Z \to X$ factors uniquely through $\tilde{X}$.*

Proof. Let $\varphi : A \to B$ be the homomorphism corresponding to f. Then, since f is dominant, φ is injective. Indeed, $f(Z) \subseteq V(\ker(\varphi))$ so if $\ker(\varphi) \neq 0$ then $f(Z)$ doesn't meet the nonempty open set $X - V(\ker(\varphi))$ (nonempty since A is a domain so (0) is prime so $(0) \in X - V(\ker(\varphi))$. So there is a unique extension of φ to a homomorphism from $\tilde{A} \to B$. Indeed, define $\overline{\varphi}(a/b) = \varphi(a)/\varphi(b)$. Then since φ is injective this is well-defined since $b \neq 0$ implies $\varphi(b) \neq 0$. Furthermore, if ψ is another possible extension of φ to $\tilde{A}$, then $\psi(b)\psi(a/b) = \psi(a) = \varphi(a) = \varphi(b)\overline{\varphi}(a/b) = \psi(b)\overline{\varphi}(a/b)$ so cancelling shows that $\psi(a/b) = \overline{\varphi}(a/b)$. Thus there is a unique morphism $f' : Z \to \tilde{X}$

whose composition with the natural map $\tilde{X} \to X$ equals f. (The natural map is induced by the inclusion $A \hookrightarrow \tilde{A}$.)

Now we prove the universal property holds when Z is an arbitrary normal integral scheme but X is still affine.

Proposition 2 *Let $X = \text{Spec } A$, $\tilde{X} = \text{Spec } \tilde{A}$ its normalization and Z be any normal integral scheme. Then every dominant morphism $f : Z \to X$ factors uniquely through $\tilde{X}$.*

PROOF. Let U_i be a cover of Z by open affines. If $U = U_i$ is any U_i then U is a normal integral affine scheme and $f|_U$ is a dominant morphism. Indeed, U is dense in Z since Z is irreducible (Proposition 3.1). Thus $f^{-1}(\overline{f(U)}) \supseteq \overline{U} = Z$ so $f^{-1}(\overline{f(U)}) = Z$ so $\overline{f(U)} \supseteq f(Z)$ so $\overline{f(U)} = \overline{f(Z)} = X$. We can thus apply the above proposition to find a unique morphism $g_i : U_i \to \tilde{X}$ such that $\psi \circ g_i = f|_U$ where $\psi : \tilde{X} \to X$. By uniqueness on a cover of $U_i \cap U_j$ by open affines, $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$. We can thus glue the morphism g_i to obtain a morphism $g : Z \to \tilde{X}$ such that $\psi \circ g = f$. The morphism g is evidently unique.

Now we can define the identification maps φ_{ij}. Let $\{U_i = \text{Spec } A_i\}$ be the open affine subsets of X. Let $\{\tilde{U}_i = \text{Spec } \tilde{A}_i\}$ be the associated normalizations. Let $\psi_i : \tilde{U}_i \to U_i$ be the morphism induced by the inclusion $A_i \hookrightarrow \tilde{A}_i$. Let $W_{ij} = \psi_i^{-1}(U_i \cap U_j)$. Then W_{ij} is an open subset of a normal scheme hence normal. $\psi_i : W_{ij} \to U_i \cap U_j \subseteq U_j$ so there is a unique morphism which we call $\varphi_{ij} : W_{ij} \to \tilde{U}_j$ such that $\psi_j|_{W_{ji}} \circ \varphi_{ij} = \psi_i|_{W_{ij}}$. By uniqueness we see that $\varphi_{ij} \circ \varphi_{ji} = id$ so $\varphi_{ij} = \varphi_{ji}^{-1}$. Furthermore, for each i, j, k, $\varphi_{ij}(W_{ij} \cap W_{ik}) = \psi_j^{-1}(\psi_i(W_{ij} \cap W_{ik})) = \psi_j^{-1}(\psi_i(W_{ij})) \cap \psi_j^{-1}(\psi_i(W_{ik})) = W_{ji} \cap W_{jk}$. By uniqueness, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $W_{ij} \cap W_{ik}$. So by the glueing lemma (Exercise 2.12) we may glue to obtain a scheme $\tilde{X}$. We can also glue the morphisms ψ_i to obtain a morphism $\psi : \tilde{X} \to X$.

Next, we must verify that the universal property holds in general. Let Z be an arbitrary normal integral scheme, and let X and $\tilde{X}$ be as above and suppose $f : Z \to X$ is a morphism. Cover X be open affines U_i. Then for each morphism $f|_{f^{-1}(U_i)}$ we can apply the above proposition to find a morphism g_i such that $\psi \circ g_i = f|_{f^{-1}(U_i)}$. By uniqueness we can glue these morphism to obtain the required morphism $g : Z \to \tilde{X}$.

Now we check that $\tilde{X}$ is a normal integral scheme. Note first that each $\tilde{U}_i$ is the spectrum of an integrally closed domain and is hence a normal integral scheme (since the localization of an integrally closed domain is integrally closed). Let $x \in \tilde{X}$. Then x is contained in some $\tilde{U}_i$. But the local ring of x in $\tilde{X}$ is the same as the local ring of x in $\tilde{U}_i$ which is integrally closed. This shows that $\tilde{X}$ is normal.

Since X is irreducible, every U_i intersects every U_j. Thus every $\tilde{U}_i$ intersects every $\tilde{U}_j$ after glueing. Since each $\tilde{U}_j$ is irreducible and they all overlap this implies $\tilde{X}$ is irreducible. Indeed, if $\tilde{X} = A \cup B$ with A and B closed, then every $\tilde{U}_i$ is either completely contained in A or in B. If they are not all contained in one of A or B then we can find an open subset U contained in A and an open subset V contained in B but not contained in A. Then $V = (U \cap V) \cup (U^c \cap V) = (A \cap V) \cup (U^c \cap V)$ which expresses V as a union of two proper closed subsets of V. $A \cap V$ is a proper subset of V since V is not contained in A and $U^c \cap V$ is a proper subset of V since $U \cap V \neq \emptyset$. This contradicts the fact that V is irreducible. Thus $\tilde{X} = A$ or $\tilde{X} = B$ whence $\tilde{X}$ is irreducible.

Now we check that the structure sheaf has no nilpotents. Let U be an open subset of $\tilde{X}$ and suppose $f \in \mathcal{O}_{\tilde{X}}(U)$ is nilpotent. Then since f is nonzero, there is some point $x \in \tilde{X}$ so that the stalk f_x of f at x in the local ring $\mathcal{O}_{\tilde{X}}$ is nonzero and nilpotent (use sheaf axiom (iii) and the definition of $\mathcal{O}_{\tilde{X}}$.) Let $\tilde{U}_i$ be some $\tilde{U}_i$ which contains x. Then the local ring at x is a localization of the integral domain $\tilde{A}_i$ so it can't contain any nilpotents. Thus the scheme $\tilde{X}$ is reduced.

Now we check that if X is of finite type over a field k, then the morphism $f : \tilde{X} \to X$ is a finite morphism. The U_i form an open cover of X and $f^{-1}(U_i) = \text{Spec } \tilde{A}_i$ is affine for each i, so we just need to check that $\tilde{A}_i$ is a finite module over A_i. This follows from Theorem 3.9A of chapter I.

Problem 4 (3.12) *Closed Subschemes of* Proj S.

(a) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to Proj T, *and the morphism $f : \text{Proj } T \to \text{Proj } S$ is a closed immersion.*

(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X =$ Proj S *defined as the image of the closed immersion* Proj $S/I \to X$. *Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geq d_0} I_d$. Show that I and I' determine the same closed subscheme.*

Proof. (a) Since φ is graded and surjective, $\varphi(S_+) = T_+$ from which it is immediate that $U = \text{Proj } T$. By the first isomorphism theorem $T \cong S/\ker(\varphi)$

so $f(\text{Proj } T) = f(\text{Proj } S/\ker(\varphi)) = V(\ker(\varphi))$ which is a closed subset of Proj S. (This is just the fact that there is a one to one correspondence between homogeneous ideals of $S/\ker(\varphi)$ and homogeneous ideals of S which contain $\ker(\varphi)$.) The map on the stalk corresponding to a point $x \in$ Proj T is the map $S_{(\varphi^{-1}(x))} \to T_{(x)}$ induced by φ. This map is surjective since φ is surjective. Thus the induced map on sheaves is surjective.

(b) Let $\varphi : S/I' \to S/I$ be the natural projection homomorphism. (This makes sense because S/I is a quotient of S/I'. Indeed, $S/I = (S/I')/\bigoplus_{0 \leq d < d_0} I_d$.) Then φ is a graded homomorphism of graded rings such that φ_d is the identity for $d \geq d_0$. So by (Exercise 2.14c) φ induces an isomorphism $f : \text{Proj } S/I \to \text{Proj } S/I'$. Since this is a morphism over Proj S (the corresponding triangle of homomorphisms commutes) it follows that I and I' give rise to the same closed subscheme.

Problem 5 (3.14) *If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.*

Proof. Since X is of finite type over k we can cover X with affine open sets $U_i = \text{Spec } A_i$ where each A_i is a finitely generated k-algebra. Let U be an open subset of X. We must show that U contains a closed point. Since the U_i cover X, U must intersect some U_i. Then $U \cap U_i$ contains a distinguised open subset of U_i. So, to show that every open set contains a closed point, it suffices to show that every nonempty distinguised open subset of each U_i contains a closed points of X. Since a distinguished open subset $(U_i)_x$ of a U_i is also the spectrum of a finitely generated k-algebra Spec $(A_i)_x$ we can just add it to our collection $\{U_i\}$. The problem thus reduces to showing that each U_i contains a closed point.

Proposition 3 *With the notation as above, if $x \in U_i$ is closed in U_i (here U_i has the subspace topology) then x is closed in X.*

Proof. Suppose $x \in U_j$. There is a natural injection $U_i \cap U_j \hookrightarrow U_j$. Let Spec $(B_i)_f$ be a distinguished open subset of U_i contained in $U_i \cap U_j$ which contains x. Then we have a morphism Spec $(B_i)_f \hookrightarrow U_j = \text{Spec } B_j$. We thus get a ring homomorphism $\varphi : B_j \to (B_i)_f$ of Jacobson rings. Since it is induced by a restriction of the identity map $X \to X$ which is a morphism over k, φ is a k-algebra homomorphism. Since $(B_i)_f$ is a finitely generated

k-algebra, $(B_i)_f$ is also a finitely generated B_j-algebra. Since x is closed in Spec $(B_i)_f$, x is a maximal ideal of $(B_i)_f$. Thus by page 132 of Eisenbud's *Commutative Algebra* $\varphi^{-1}(x)$ is a maximal ideal of B_j. Thus x is also a closed point of U_j in the subspace topology on U_j. Thus $X - x = \cup_i(U_i - x)$ is a union of open subsets of X, hence open in X, so x is closed.

To finish we just need to know that U_i has a closed point.

Proposition 4 *Let $X =$ Spec A be an affine scheme with A a finitely generated k-algebra. Then any nonempty distinguished open subset of X contains a closed point.*

PROOF. The key observation is that A is a Jacobson algebra since it finitely generated over a field, so by page 131 of Eisenbud's *Commutative Algebra* the Jacobson radical of A equals the nilradical of A. Let $D(f)$ be a nonempty distinguised open subset of X. Then some prime omits f so f is not in the nilradical of A. Thus f is not in the Jacobson radical of A so there is some maximal ideal m so that $f \notin m$. Then $m \in D(f)$ and m is a closed point of X since m is maximal.

Strangely enough I never used the hypothesis that X is of finite type over k but just the weeker hypothesis that X is *locally* of finite type over k. Did I miss something?

Finally, we present a counterexample in the more general situation. Let $X =$ Spec $Z_{(2)}$. Then X contains precisely one closed point, the ideal (2). So the set of closed points in X is not dense in X. In fact, if X is the spectrum of any DVR we also get a counterexample.

Problem 6 (4.2) *Let S be a scheme, let X be a reduced scheme over S, and let Y be a seperated scheme over S. Let f_1 and f_2 be two S-morphisms of X to Y which agree on an open dense subset U of X. Show that $f_1 = f_2$. Give examples to show that this result fails if either (a) X is nonreduced, or (b) Y is nonseparated.*

PROOF. Let $g = (f_1, f_2)_S : X \to Y \times_S Y$ be the product of f_1 and f_2 over S. By hypothesis the diagonal $T = \Delta_Y(Y)$ is a closed subscheme of $Y \times_S Y$. Thus $Z = g^{-1}(T)$ is a closed subscheme of X. If $h : U \to Y$ is the common restriction of f_1 and f_2 to U, then, since $g|_U$ makes the correct diagram commute, the restriction of g to U is $g' = (h, h)_S$ and $g' = \Delta_Y \circ h$ since $\Delta_Y \circ h$ makes the correct diagram commute.

Thus $\Delta_Y^{-1}(T) = Y$ implies

$$g^{-1}(T) \supseteq (g')^{-1}(T) = h^{-1}(\Delta_Y^{-1}(T)) = h^{-1}(Y) = U.$$

Thus $g^{-1}(T)$ is a closed set which contains the dense set U. Thus $g^{-1}(T) = X$ so $g(X) \subseteq T$. So, by the proposition below, since X is reduced, g factors as $g = \Delta_Y \circ f$ where $f : X \to Y$. From the definition of Δ_Y, we see that $\pi_1 \circ \Delta_Y = \mathrm{id}_Y = \pi_2 \circ \Delta_Y$. Thus $f_1 = \pi_1 \circ g = \pi_1 \circ \Delta_Y \circ f = f$ and $f_2 = \pi_2 \circ g = \pi_2 \circ \Delta_Y \circ f = f$ so $f = f_1 = f_2$, as desired.

Proposition 5 *Let X be a reduced scheme, $f : X \to Y$ a morphism, Z a closed subscheme of Y, $j : Z \hookrightarrow Y$, such that $f(X) \subseteq j(Z)$. Then f factors uniquely as*

$$X \xrightarrow{g} Z \xhookrightarrow{j} Y.$$

Proof. First assume X and Y are affine, $X = \mathrm{Spec}\ A$, $Y = \mathrm{Spec}\ B$, $Z = \mathrm{Spec}\ B/I$. (Use exercise 3.11 to see that every closed subscheme Z of Y is of the form Spec B/I.) Let $\varphi : B \to A$ be the homomorphism which induces f. Since $f(X) \subseteq Z$, the inverse image of any prime of A contains I. Since A is reduced the intersection of all primes of A equals $\{0\}$. Thus

$$\ker(\varphi) = \varphi^{-1}(\{0\}) = \varphi^{-1}(\cap_{\mathrm{primes} p} p) \subseteq I$$

so φ factors uniquely through B/I.

$$B \xrightarrow{j^\#} B/I \xhookrightarrow{g^\#} A$$

This proves the proposition when X and Y are affine.

Now suppose X is an arbitrary reduced scheme. Cover X by open affines $X = \cup_i U_i$. For each i let g_i be the unique map which factors $f|_{U_i}$ through Z. By uniqueness $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$, so we can glue the g_i to obtain a morphism $g : X \to Z$ such that $j \circ g = f$. Now suppose both X and Y are arbitrary. Cover Y by open affines, take their inverse images in X, perform the construction locally for each one, use uniqueness and glue.

Counterexamples.

(a) Let $A = k[x, y]/(x^2, xy)$, let $X = Y = \mathrm{Spec}\ A$ and let $S = \mathrm{Spec}\ k$. Then Y is affine hence seperable over S, but X is not reduced. Let $f : X \to Y$ be the morphism induced by the identity homomorphism $\mathrm{id} : A \to A$. Let $g : X \to Y$ be the morphism induced by the homomorphism $\varphi : A \to A$:

$x \mapsto 0, y \mapsto y$. Let $U = D(y) = \text{Spec } A_y$. Then since $A_y \cong \text{Spec } k[y, y^{-1}]$, the localized homomorphisms agree, $\text{id}_y = \varphi_y$. Thus $f|_U = g|_U$. Now X is irreducible since A has just one minimal prime, namely (x), so U is dense in X. But, $f \neq g$. since $f^\# = \text{id} \neq \varphi = g^\#$.

(b) Let X be the affine line and Y the affine line with a doubled origin both thought of as schemes over $S = \text{Spec } k$. Let $f_1 : X \to Y$ be one of the inclusions of the affine line in Y and let $f_2 : X \to Y$ be the other one. Then f_1 and f_2 agree on X minus the origin but not on X.

[Reference, EGA, I.8.2.2.1.]

Problem 7 (4.4) *Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure is proper over S.*

PROOF. First we show that since X, Y and Z are of finite type over S and S is noetherian, X, Y and Z are noetherian. Suppose $g : X \to S$ is the map from X to S. Cover S by finitely many Spec A_i, A_i noetherian. Then for each $f^{-1}(\text{Spec } A_i) = \cup_j \text{Spec } B_{ij}$, with B_{ij} a finitely generated A_i-algebra. Since A_i is noetherian each B_{ij} is noetherian (this is the Hilbert basis theorem). Since $X = \cup_{ij} \text{Spec } B_{ij}$, X is noetherian. One shows that Y and Z are noetherian in exactly the same way.

Since the following diagram commutes 4.8(e) implies $f|_Z$ is proper.

Thus $f|_Z(Z)$ is closed in Y. (I'm assuming Z is an S-subscheme of X so that the diagram must commute.)

We now have $f(Z) \hookrightarrow Y \to S$. We must show the composition $f(Z) \to S$ is proper. By Corollary 4.8a the closed immersion $f(Z) \hookrightarrow Y$ is proper. We are given that the map $Y \to S$ is seperated and of finite type. Since the composition of seperated morphisms is seperated and the composition of finite type morphisms is of finite type, the morphism $f(Z) \to S$ is seperated and of finite type. The hard part is to show that it is universally closed.

Since the morphism $Z \to S$ is proper it is closed and from above the morphism $Z \to f(Z)$ is closed so the morphism $f(Z) \to S$ is closed. Let W be an any scheme over S. We must show that the map $f(Z) \times_S W \to W$ is closed. We have the following diagram.

If we can show that g_1 is surjective we will be done. For then if A is closed subset of $f(Z) \times_S W$,

$$g_2(A) = g_3(g_1^{-1}(A)),$$

which, since g_3 is closed, is also closed. (We wouldn't have equality in the above expression if g_1 weren't surjective.)

In order to establish the surjectivity of g_1 we prove that the property of being a surjective morphism is preserved under base extension. It will then follow, since g_1 is a base extension of the surjective morphism $Z \to f(Z)$, that g_1 is surjective.

Proposition 6 *Let X and Y be schemes over S. Suppose $x \in X$ and $y \in Y$ both lie over the same point $s \in S$. Then there exists $\alpha \in X \times_S Y$ such that $p_X(\alpha) = x$ and $p_Y(\alpha) = y$.*

Proof. Let $g_1 : \text{Spec } (k(x)) \to X$ and $g_2 : \text{Spec } (k(y)) \to Y$ be the natural maps with $g_1((0)) = x$ and $g_2((0)) = y$. Let $Z = \text{Spec } (k(x)) \times_{\text{Spec } (k(s))} \text{Spec } (k(y)) = \text{Spec } (k(x) \otimes_{k(s)} k(y))$, and let π_1 be the projection to $\text{Spec } (k(x))$, π_2 the projection to $\text{Spec } (k(y))$. Let $g = g_1 \times_S g_2$ be the product of $g_1 \circ \pi_1$ with $g_2 \circ \pi_2$. So $g : Z \to X \times_S Y$. See the following diagram.

Since Z is the spectrum of the tensor product of two fields over a common base field, $Z \neq \emptyset$ so there is some $z \in Z$. By the definition of g, $g_1 \circ \pi_1 = p_x \circ g$ and $g_2 \circ \pi_2 = p_y \circ g$ so $x = g_1 \circ \pi_1(z) = p_x \circ g(z)$ and $y = g_2 \circ \pi_2(z) = p_y \circ g(z)$ so we may take $\alpha = g(z)$.

Proposition 7 *If $f : X \to Y$ is a surjective S-morphism then $f \times 1 : X \times_S S' \to Y \times_S S'$ is surjective.*

PROOF. We have the following diagram.

Let $y' \in Y \times_S S'$. Then $q(y') \in Y = f(X)$ so there is $x \in X$ such that $f(x) = q(y')$. Then by the above proposition there is some $\alpha \in X \times_S S'$ such that $p(\alpha) = x$ and $(f \times 1)(\alpha) = y'$. Thus $f \times 1$ is surjective.