Algebraic Geometry Homework
II.5, 1,2,3,4,5,7,8

William A. Stein

April 4, 1996

Problem 1 (II.5.1) *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathcal{E}$, denoted by $\check{\mathcal{E}}$, to be the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.*

(a) Show that $(\check{\mathcal{E}})\check{} \cong \mathcal{E}$.

(b) For any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{F}$.

(c) For any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}))$.

(d) (Projection Formula). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}$.

PROOF. (a) A free module of finite rank is *canonically* isomorphic to its double-dual via $\check{m}(\lambda) = \lambda(m)$ where $m \in M$, $\lambda \in \check{M}$, and $\check{m} \in (\check{M})\check{}$. Let U be an open set on which $\mathcal{E}|U$ is a free $\mathcal{O}_X$-module of finite rank. Define a map $\varphi : \mathcal{E}|_U \to (\check{\mathcal{E}})\check{}|_U$ by, for all $V \subseteq U$, $\mathcal{E}(V) \to (\check{\mathcal{E}})\check{}(V)$ is the isomorphism described above. Since the isomorphisms are canonical, we can patch on intersections and define a global isomorphism.

(b) As above we may assume $\mathcal{E}$ is a free $\mathcal{O}_X$-module. Let $e_1, \ldots, e_n$ be a basis for $\mathcal{F}$ and let $e_1^*, \ldots, e_n^*$ be the corresponding dual basis. Let U be an open subset of X. Define $\varphi_U : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})|_U \to (\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})|_U$ by $f \mapsto \sum_{i=1}^n e_i^* \otimes f(e_i)$. Define $\psi_U : (\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})|_U \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})|_U$ by $f \otimes a \mapsto (x \mapsto f(x)a)$. For convenience of notation write $\varphi = \varphi_U$ and $\psi = \psi_U$. Let $f \otimes a \in (\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})|_U$. Then $\varphi \circ \psi(f \otimes a) = \varphi(x \mapsto f(x)a) = \sum_{i=1}^n e_i^* \otimes f(e_i)a = \sum_{i=1}^n e_i^* f(e_i) \otimes a = f \otimes a$. Let $f \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$, then $\psi \circ \varphi(f) = \psi(\sum_{i=1}^n e_i^* \otimes f(e_i)) = (x \mapsto \sum_{i=1}^n e_i^*(x) f(e_i)) = (x \mapsto$

$f(\sum_{i=1}^n e_i^*(x)e_i)) = (x \mapsto f(x))$. Thus φ and ψ are inverse bijective homomorphisms, hence ring isomorphisms, and since they respect the restriction maps we see that the corresponding sheaves are isomorphic.

(c) On each open set define $\varphi|_U : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}\otimes\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, sg))$ by $f \mapsto (a \mapsto (e \mapsto f(e \otimes a)))$. (For notational convenience we omit the sheaf restrictions.) If $\varphi(f) = 0$ then the map $(a \mapsto (e \mapsto f(e \otimes a)))$ is 0 so f is the zero map, hence φ is injective. Let $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, sg))$. Define $g \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G})$ by $g(a \otimes b) = (f(b))(a)$. Then $\varphi(g) = (a \mapsto (e \mapsto g(e \otimes a))) = (a \mapsto (e \mapsto (f(a))(e)) = (a \mapsto f(a)) = f$ so φ is surjective. Thus φ is the desired isomorphism which, since φ evidently commutes with the restriction maps, induces an isomorphism of sheaves.

(d) First we consider the case when $\mathcal{E} \cong \mathcal{O}_Y^n$. One one hand,

$$\begin{aligned} f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E} &\cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^n \\ &\cong \oplus_{i=1}^n (f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y) \\ &\cong \oplus_{i=1}^n f_*(\mathcal{F}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E} &= \mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y^n) \\ &\cong \mathcal{F} \otimes_{\mathcal{O}_X} (f^{-1}(\mathcal{O}_Y^n) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &\cong \mathcal{F} \otimes_{\mathcal{O}_X} ((f^{-1}(\mathcal{O}_Y)^n \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &\cong \mathcal{F} \otimes_{\mathcal{O}_X} (\sum_{i=1}^n f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &\cong \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X^n) \\ &\cong (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X)^n = \mathcal{F}^n \end{aligned}$$

where $f^{-1}(\mathcal{O}_Y^n) \cong f^{-1}(\mathcal{O}_Y)^n$ since f^{-1} is a left adjoint functor hence commutes with direct sums (which are a right universal construction).

Putting this together we have that

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*(\mathcal{E})) = f_*(\mathcal{F}^n) = f_*(\oplus_{i=1}^n \mathcal{F}) = \oplus_{i=1}^n f_*(\mathcal{F}).$$

In general, we construct isomorphisms as above on an open cover then, since all of the isomorphisms are canonical, the isomorphisms match up on the intersections so we can glue to obtain an isomorphism.

Problem 2 (II.5.2) *Let R be a discrete valuation ring with quotient field K, and let $X = Spec R$.*

(a) To give an $\mathcal{O}_X$-module $\mathcal{F}$ is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$.

(b) That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

Proof. (a) First suppose we are given an $\mathcal{O}_X$-module $\mathcal{F}$. Since R is a DVR, X has exactly two nonempty open sets, X and the set consisting of the generic point, $\{\xi\}$. Let $M = \Gamma(\mathcal{F}, X)$ and let $L = \Gamma(\mathcal{F}, \{\xi\})$. Since $\Gamma(\mathcal{O}_X, X) = R$ and $\Gamma(\mathcal{O}_X, \{\xi\}) = K$, M is an R-module and L is a K-vector space. Let $g : M \to L$ be the restriction map. Define ρ by $\rho(m \otimes \alpha) = \alpha \cdot g(m)$. Since g is a homomorphism, so is ρ.

Now suppose we are given an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$. Define an $\mathcal{O}_X$-module $\mathcal{F}$ as follows. Let $\Gamma(\mathcal{F}, X) = M$ and $\Gamma(\mathcal{F}, \{\xi\}) = L$. Define the restriction map $g : M \to L$ by $g(m) = \rho(m \otimes 1)$. We just need to check that g is a valid restriction map. Let $r \in R, m \in M$, then g is a valid restriction map iff $r \cdot g(m) = g(rm)$, that is, when $r \cdot \rho(m \otimes 1) = \rho(rm \otimes 1) = \rho(m \otimes r)$. So we must verify that the given homomorphism ρ is R-linear. But there is absolutely no reason why this should be the case! (For example, let $M = L = K$, then $\rho : K \to K$ and it is easy to construct nontrivial homomorphisms of the additive group of a field). I think the problem is imprecisely stated. It should be assumed throughout that ρ is K-linear.

(b) First suppose $\mathcal{F}$ is quasi-coherent. Then $\mathcal{F} = \tilde{M}$ so Proposition 5.1 implies that $L = \Gamma(\tilde{M}, \{\xi\}) = (R - 0)^{-1}M \cong M \otimes_R K$. Thus ρ must be an isomorphism. Conversely, if ρ is an isomorphism, we see that $\mathcal{F} \cong \tilde{M}$ since they are the same on each open set and the restriction map is the same.

Problem 3 (II.5.3) *Let $X = Spec A$ be an affine scheme. Show that the functors $\tilde{}$ and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, there is a natural isomorphism*

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}).$$

Proof.

Define a homomorphism $F : \mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \to \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$ as follows. Send a ring homomorphism $\varphi : M \to \Gamma(X, \mathcal{F})$ to the morphism of sheaves $F(\varphi) : \tilde{M} \to \mathcal{F}$. It suffices to define $F(\varphi)$ on distinguished open sets (Eisenbud & Harris, page 13). For $f \in A$ let $F(\varphi)_{D(f)}$ be the map

$$\frac{m}{f^n} \mapsto \frac{1}{f^n} \cdot \mathrm{res}_{X,D(f)}(\varphi(m))$$

where $\mathrm{res}_{X,D(f)} : \mathcal{F}(X) \to \mathcal{F}(D(f))$ is the restriction map of $\mathcal{F}$. $F(\varphi)_{D(f)}$ is a well-defined homomorphism since both φ and $\mathrm{res}_{X,D(f)}$ are homomorphisms and φ is an A-module homomorphism. Next note that $F(\varphi)$ commutes with the restriction maps since each $\mathrm{res}_{X,D(f)}$ does. To see that F is injective suppose φ and ψ are two homomorphisms $M \to \Gamma(X, \mathcal{F})$. If $F(\varphi) = F(\psi)$ then, in particular, $\varphi = F(\varphi)_X = F(\psi)_X = \psi$. To see that F is surjective, let $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$. Define $\psi : M \to \Gamma(X, \mathcal{F})$ by letting $\psi = \varphi_X$, that is, by taking the induced map on global sections. Then, for $f \in A$, $F(\psi)_{D(f)} : \tilde{M}(D(f)) \to \mathcal{F}(D(f))$ is the map $(\frac{m}{f^n} \mapsto \frac{1}{f^n}\mathrm{res}_{X,D(f)} \circ \varphi_X(m))$ which, since φ is a morphism of sheaves, equals $= (\frac{m}{f^n} \mapsto \frac{1}{f^n}\varphi_{D(f)}(m) = \varphi_{D(f)}(\frac{m}{f^n}) = \varphi_{D(f)}$. (We are just using the fact that φ commutes with the appropriate restriction maps.) Thus $F(\psi)$ agrees with φ on a basis for X hence $F(\psi) = \varphi$. This shows that F is surjective. So F is an isomorphism, as required.

Problem 4 (II.5.4) *Show that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathcal{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathcal{F}$ is coherent iff it is locally a cokernel of a morphism of free sheaves of finite rank.*

Proof. Suppose first that $\mathcal{F}$ is quasi-coherent. Let $x \in X$. Then there is an affine open neighborhood $U = Spec A$ of x such that $\mathcal{F}|_U \cong \tilde{M}$, M an A-module. It suffices to show that M is isomorphic to a cokernel of a morphism of finitely generated A-algebras. Indeed, if $\varphi : A^{(I)} \to A^{(J)}$ then $\mathrm{Coker}(\tilde{\varphi}) = \tilde{A^{(J)}}/\varphi(A^{(I)})\tilde{\ } = (A^{(J)}/\varphi(A^{(I)}))\tilde{\ }$ since, for all $f \in A$, $(A^{(J)})_f/\varphi(A^{(I)})_f = (A^{(J)}/\varphi(A^{(I)}))_f$ so that they agree on a basis.

Let $A^{|M|}$ be the free A-module on the elements of M. Let $\varphi : A^{|M|} \to M$ be the natural map. Similiary, let $\psi : A^{|\ker(\varphi)|} \to \ker(\varphi) \subseteq A^{|M|}$ be the natural map. Then $\mathrm{Coker}(\psi) \cong A^{|M|}/\ker(\varphi) \cong M$, as required.

Now assume the $\mathcal{F}$ is coherent. We proceed as above but now M is a finitely generated A-module, generated by $e_1, \ldots, e_n$, say, and we must show that M is the cokernel of a morphism of free modules of finite rank. Let

$\varphi : A^n \to M$ be the map which takes the ith generator $(0, \ldots, 1, \ldots, 0)$ of A^n to $e_i \in M$. Then $\ker(\varphi)$ is a submodule of M so, since M is neotherian (any finitely generated module over a noetherian ring is noetherian), $\ker(\varphi)$ is finitely generated. Let $f_1, \ldots, f_m$ be a generating set. Let $\psi : A^m \to \ker(\varphi)$ be the surjection defined by sending the ith basis element of A^m to f_i. Then $\mathrm{Coker}(\psi) \cong A^n/\psi(A^m) \cong A^n/\ker(\varphi) \cong M$. Thus M is isomorphic to a cokernel of a morphism of free sheaves of finite rank.

Problem 5 (II.5.5) *Let $f : X \to Y$ be a morphism of schemes.*

(a) Show by example that if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.

(b) Show that a closed immersion is a finite morphism.

(c) If f is a finite morphism of neotherian schemes, and if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ is coherent on Y.

PROOF. (a) Lex k be a field, let $X = Spec(k[x]_x)$, $Y = Spec(k[x])$ and $\mathcal{F} = \mathcal{O}_X$. Then $f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$, so $f_*(\mathcal{F}) = (k[x]_x)\tilde{}^Y$. But $(k[x]_x)\tilde{}^Y$ is not a coherent sheaf of $\mathcal{O}_Y$-modules. Indeed, if it were, there would be a distinguished neighborhood $D(f)$ of 0 so that $(k[x]_x)\tilde{}^Y|_{D(f)}$ is a finitely generated module over $k[x]$ (here I'm using the fact that over open set is distinguished in the topology of $Spec(k[x])$). But, $k[x]_f$ is never a finitely generated module over $k[x]$ for any f of degree at least 1. For $k[x]_f$ contains elements of arbitrarily small degree whereas the degrees of elements of $k[x]\{\alpha_1, \ldots, \alpha_n\}$ are bounded below. ($k[x]\{\alpha_1, \ldots, \alpha_n\}$ is the $k[x]$-module generated by $\alpha_1, \ldots, \alpha_n$.)

(b) Let $f : Y \to X$ be a closed immersion. Let $U = Spec(A) \subseteq X$ and let $W = f^{-1}(U) \subseteq Y$. Then W is a a scheme (give it the induced scheme structure as an open subset of Y). Furthermore, $f(W) = U \cap f(Y)$ is a relatively closed subset of U, that is, a closed subset of $Spec(A)$. The map $f^\# : \mathcal{O}_U \to \mathcal{O}_W$ is surjective since $f^\# : \mathcal{O}_X \to \mathcal{O}_Y$ is surjective and surjectivety is a local property. Thus W is a closed subscheme of $Spec(A)$ so, by Corollary 5.10, $W \cong Spec(A/I)$ for some ideal I of A. Since A/I is a finitely generated A module (generated by $1 + I$), f is a finite morphism.

(c) Let $U = Spec(A) \subseteq Y$ be an affine open subset of Y. Then, since f is finite, $f^{-1}(U) = Spec(B)$ is affine with B is a finitely generated A-module. By Proposition 5.4 it suffices to show that $f_*(\mathcal{F})|_U$ is $\tilde{M}$ for some finitely generated A-module M. Now by Proposition 5.4, since $f^{-1}(U)$ is affine and B is noetherian, $\mathcal{F}|_{f^{-1}(U)} = \tilde{M}$ for some finitely generated B module M.

But $f_*(\mathcal{F})|_U = (f|_{f^{-1}(U)})_*\mathcal{F}|_{f^{-1}(U)} = (f|_{f^{-1}(U)})_*(\tilde{M}) = (\tilde{A^M})$ where the last equality follows from Proposition 5.2(d). Since B is a finite module over A and M is a finite module over B, it follows that M is a finite module over A which completes the proof.

Problem 6 (II.5.7) *Let X be a noetherian scheme, and let sf be a coherent sheaf.*

(a) If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_X$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.

(b) $\mathcal{F}$ is locally free iff its stalks $\mathcal{F}_x$ are free $\mathcal{O}_X$-modules for all $x \in X$.

(c) $\mathcal{F}$ is invertible iff there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$.

Proof. (a) Let $U = Spec(A)$ be a neighborhood of x so that $\mathcal{F}|_U = \tilde{M}$, M a finitely generated A-module. Then $\mathcal{F}_x = M_x$ so we have reduced the problem to the following purely algebraic result.

Proposition 1 *If M is a finitely generated free A module and there is a prime $\wp$ of A such that $M_\wp$ is a free $A_\wp$-module, then there exists $f \in A$ such that $f \notin \wp$ and M_f is a free A_f-module.*

Once we have proven this we will know that $\mathcal{F}|_{D_U(f)}$ is free since Proposition 5.1 asserts that $\tilde{M}(D(f)) \cong M_f$.

Proof. (of Proposition) Let $a_1, \ldots, a_n \in M$ be a free basis of $M_\wp$ over $A_\wp$ (we can clear denominators so that we may assume all a_i lie in M.) Let $b_1, \ldots, b_m \in M$ be a generating set for M over A. For each i we can write b_i as an $A_\wp$-linear combination of the a_i. Clearing denominators we see that $d_i b_i \in A\{a_1, \ldots, a_n\}$ for some $d_i \notin \wp$. Let $f = \prod d_i$, then $f \notin \wp$ and $a_1, \ldots, a_n$ have A_f-span including all of the b_i, and thus including M, and thus including M_f. But $a_1, \ldots, a_n$ is free over $A_\wp$ hence over A_f since $A_f \subseteq A_\wp$. [This proposition is in Bourbaki, *Commutative Algebra*, II.5.1, although the proof is more abstract than mine.]

(b) ($\Longrightarrow$) Let $x \in X$ and let $U = Spec(A)$ be an open neighborhood of x such that $\mathcal{F}|_U = \tilde{M}$ with M a free A module. Suppose $\wp$ is the prime of A corresponding to x. Then $\mathcal{F}_x = M_\wp$ which is a free $A_\wp$-module. Indeed, if $e_1, \ldots, e_n$ is a free A-basis for M, then it is also a free $A_\wp$-basis for $M_\wp$. For if $\frac{a_1}{b_1}e_1 + \cdots + \frac{a_n}{b_n}e_n = 0, b_i \notin \wp$, then $a_1\frac{b}{b_1}e_1 + \cdots + a_n\frac{b}{b_n}e_n = 0, b = \prod b_i$ so $b\frac{a_i}{b_i} = 0$ for each i, so $\frac{a_i}{b_i} = 0$ in the localization $A_\wp$ since $b \notin \wp$.

($\Longleftarrow$) By part (a) every point has a neighborhood on which $\mathcal{F}$ is free. Therefore X can be covered by open affines on which $\mathcal{F}$ is free so $\mathcal{F}$ is locally free.

(c) ($\Longrightarrow$) Let $\mathcal{G} = \check{\mathcal{F}} = \mathcal{H}\text{om}(\mathcal{F}, \mathcal{O}_X)$, we will show that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. To define a morphism $\varphi : \mathcal{F} \otimes \mathcal{H}\text{om}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{O}_X$ it is enough to define φ on the presheaf $(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{H}\text{om}(\mathcal{F}(U), \mathcal{O}_X(U))$. Define φ by $a \otimes f \mapsto f(a)$. Thus φ commutes with the restrictions so φ defines a valid morphism of sheaves.

Let U be an open affine subset of X such that $\mathcal{F}(U)$ is a free $\mathcal{O}_X(U)$-module of rank 1 with basis e_0. Then $\mathcal{H}\text{om}(\mathcal{F}(U), \mathcal{O}_X(U))$ has basis e_0^* where $e_0^*(e_0) = 1$. Thus $\mathcal{H}\text{om}(\mathcal{F}(U), \mathcal{O}_X(U))$ is a free $\mathcal{O}_X(U)$-module of rank 1. Thus every element of $\mathcal{F}(U) \otimes \text{Hom}(\mathcal{F}(U), \mathcal{O}_X(U))$ can be written in the form $a \otimes f$ (as opposed to as a sum of such products). Now $\varphi_U(a \otimes f) = 0$ implies $f(a) = 0$ which implies $a = 0$ or $f = 0$ so $a \otimes f = 0$, so φ_U is injective. Since $\varphi_U(ae_0 \otimes e_0^*) = a$, φ_U is surjective. Thus φ_U is an isomorphism.

Use the definition of locally free of rank 1 to cover X by affine open sets U such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$ module of rank 1. Then, by Proposition 5.1 (c) and an argument like that for (b) above, any distinguished open subset of U has a $\mathcal{F}$-sections free of rank 1. Since φ is an isomorphism on each of these distinguished open sets (use the argument in the above paragraph) and these distinguished open sets form a basis for the topology on X it follows that φ must be an isomorphism.

($\Longleftarrow$) Because of part (b) above it suffices to show that $\mathcal{F}_x$ is free of rank one for each $x \in X$. Since $\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{G}_x = (\mathcal{F} \otimes \mathcal{G})_x \cong \mathcal{O}_{X,x}$ the problem is reduced to the following purely algebraic statement.

Proposition 2 *Let M and N be finitely generated modules over a local ring (A, m) and suppose that $M \otimes_A N \cong A$. Then M is free of rank 1.*

Proof. Let $k = A/m$. Let $\nu : M \otimes_A N \to A$ be the given isomorphism. Then, taking the product with the identity, we get an isomorphism $\nu \otimes 1_k : (M \otimes_A N) \otimes_A k \to A \otimes_A k \cong k$ (it is obvious that $\nu \otimes 1_k$ is surjective, but it is not at all obvious that it is injective, for this see the Bourbaki reference below.) Thus $k \cong M \otimes_A N \otimes_A k = M \otimes_A (k \otimes_A k) \otimes_A N = (M \otimes_A k) \otimes_A (N \otimes_A k) = (M \otimes_A k) \otimes_k (N \otimes_A k) = (M/mM) \otimes_k (N/mN)$ so, since the k-rank of $(M/mM) \otimes_k (N/mN)$ is 1 and is the product of the ranks of M/mM and N/mN, each has rank 1. In particular, M/mM is monogeneous (generated by one element) as an A/m-module and hence as an A-module so, by Nakayama's lemma, M is monogeneous as an A-module. Since $\text{Ann}_A(M)$ anihilates $M \otimes_A N = A$ as well, it is 0 (any element of $\text{Ann}_A(M)$ would have to annihilate the identity of A and hence be 0). Thus

M is a free module of rank one over A. [See Bourbaki, *Commutative Algebra*, II.5.4, for a more general theorem.]

Problem 7 (II.5.8) *Again let X be a noetherian scheme, and $\mathcal{F}$ a coherent sheaf on X. We will consider the function*

$$\varphi(x) = \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_X} k(x)$$

where $k(x) = \mathcal{O}_X/m_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

(a) The function φ is upper semi-continuous, i.e., for any $n \in \boldsymbol{Z}$, the set $\{x \in X : \varphi(x) \geq n\}$ is closed.

(b) If $\mathcal{F}$ is locally free, and X is connected, then φ is a constant function.

(c) Conversely, if X is reduced, and φ is constant, then $\mathcal{F}$ is locally free.

Proof.

(a) We must show that $\{x : \varphi(x) \geq n\}$ is closed. A good way to do this is by showing that $\{x : \varphi(x) < n\}$ is open. To do this we show that if $\varphi(x) = m$ then there is an open neighborhood U of x so that, for all $y \in U$, $\varphi(y) \leq m$. Since we need only look locally, we can assume that $X = Spec A$, $\mathcal{F} = \tilde{M}$, M a finitely generated A-module. Note that $\mathcal{F}_x \otimes_{\mathcal{O}_X} k(x) = M_\wp \otimes_{A_\wp} A_\wp/\wp A_\wp = M_\wp/\wp M_\wp$. Let $s_1, \ldots, s_m \in M$ be elements whose images form a basis for the vector space $M_\wp/\wp M_\wp$ over $A_\wp/\wp A_\wp$ (to do this choose a basis for $M_\wp/\wp M_\wp$ then clear fractions). Note that the images of the s_i in fact generate $M_\wp/\wp M_\wp$ as an $A_\wp$-module. By Nakayama's lemma the s_i generate $M_\wp$ as an $A_\wp$-module. Let $m_1, \ldots, m_k$ be a generating set for M over A. Write $m_j = \sum \frac{a_i}{b_i} s_i$, $b_i \notin \wp$, then, if $c_j = \prod b_i$, $c_j m_j$ is in the A-span of the s_i. Let $f = \prod c_j$. Then $\wp \in D(f)$ and if $q \in D(f)$, then $m_1, \ldots, m_k$ all lie in the A_q-span of $s_1, \ldots, s_m$ (since $c_j m_j$ is in the A-span of the s_i and c_j is inverted in A_q. Thus M is spanned by the s_i over A_q, so M_q is spanned by the s_i over A_q. It follows that $\varphi(q) = \dim M_q/qM_q \leq m$ since the images of the s_i generate M_q/qM_q as a vector space over $A_q/qM_q = k(q)$. Taking $D(f)$ as our open neighborhood completes the proof.

(b) Choose n so that some section of $\mathcal{F}$ has rank n. Let U be the union of all open sets W such that $\mathcal{F}|_W \cong \mathcal{O}_X^n|_W$. Then U is nonempty. Let V be the union of all open sets W such that $\mathcal{F}|_W \cong \mathcal{O}_X^m|_W$, $m \neq n$. Since $\mathcal{F}$ is locally free, $U \cup V = X$. Suppose $x \in U \cap V$, then $\mathcal{F}_x$ has rank n and rank $m \neq n$ (since rank is preserved under localization), a contradiction. Thus $U \cap V = \emptyset$. Since U is nonempty and open, $X - U = V$ is open and X is connected, thus we conclude that $V = X - U = \emptyset$. Thus every point is contained in an open set W such that $\mathcal{F}|_W \cong \mathcal{O}_X^n|_W$.

Let $x \in X$ and let $U = Spec(A)$ be an affine open set containing x such that $\mathcal{F}|_U \cong \tilde{M}$. By the above argument, M is a free A-module of rank n. Thus $\varphi(x) = \dim_k M_x \otimes_{A_x} A_x/m_x = \dim_k A_x^n \otimes_k A_x/m_x = \dim_k (A_x/m_x)^n = \dim_k k^n = n$, as desired.

(c) Let $x \in X$. By exercise 5.7b it suffices to show that the stalk $\mathcal{F}_x$ is free. Since $\mathcal{F}$ is coherent we can find an affine open set $U = Spec(A)$ such that $\mathcal{F}|_U = \tilde{M}$ for some finitely generated A-module, $x \in U$, and A_f is reduced for each $f \in A$. Let $\wp$ be the prime of A corresponding to x. We must show that $M_\wp$ is free over $A_\wp$. Let $s_1, \ldots, s_n \in M$ be preimages of a basis of $M_\wp/\wp M_\wp$ over $k(x) = A_\wp/\wp A_\wp$ (find these as in part (a)). Then, by Nakayama's lemma, the s_i generate $M_\wp$ over $A_\wp$.

We must show that the s_i are linearly independent over $A_\wp$. It will then follow that $M_\wp$ is free of rank n over $A_\wp$. So suppose

$$\frac{a_1}{b_1}s_1 + \cdots + \frac{a_n}{b_n}s_n = 0$$

in $M_\wp$ with $\frac{a_i}{b_i} \in A_\wp$. Then for each i, $b_i \notin \wp$ and $a_i \in A$. Since the s_i are linearly independent over $A_\wp/\wp A_\wp$, for each i there exists $c_i \notin \wp$ such that $\frac{c_i a_i}{b_i} \in \wp$. Thus $c_i a_i \in \wp$ so $a_i \in \wp$. Let r be as in part (a) so that $q \in D(r)$ implies the s_i generate at least M over A_q. By definition there exists $c \in A$ such that

$$c(b_2 \cdots b_n a_1 s_1 + \cdots + b_1 \cdots b_{n-1} a_n s_n) = 0$$

in A. Let $f = rc\prod b_i$, then if $q \in D(f)$, then $s_1, \ldots, s_n$ generate M_q/qM_q over A_q and, since M_q/qM_q has dimension n (since φ is constant), the s_i are actually a basis for M_q/qM_q over A_q/qA_q.

Since $c|f$, $c \notin q$ so, as above, $\frac{a_1}{b_1}s_1 + \cdots + \frac{a_n}{b_n}s_n = 0$ in M_q, so, as above, $a_i \in q$ for each i. Thus, for all $q \in D(f)$, $a_i \in q$, so a_i lies in the nilradical of A_f which, since A_f is reduced, means that $a_i = 0$ in A_f. So a_i maps to 0 under the map $A_f \to A_\wp$. Thus $s_1, \ldots, s_n$ are linearly independent over $A_\wp$ so $M_\wp$ is free of rank n over $A_\wp$. Applying exercise (5.7b) then completes the proof.