Homework 2, MAT256B
Chapter III, 4.8, 4.9, 5.6

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1 Homework

Exercise 1.1. **(4.8)** *Cohomological Dimension.* Let X be a neotherian separated scheme. We define the *cohomological dimension* of X, denoted $\mathrm{cd}(X)$, to be the least integer n such that $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\mathrm{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\mathrm{cd}(X) \leq \dim X$.

(a) In the definition of $\mathrm{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X.

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X.

(c) Suppose X has a covering by $r + 1$ open affine subsets. Use Čech cohomology to show that $\mathrm{cd}(X) \leq r$.

(d) If X is a quasi-projective variety of dimension r over a field k, then X can be covered by $r + 1$ open affine subsets. Conclude that $\mathrm{cd}(X) \leq \dim X$.

(e) Let Y be a set-theoretic complete intersection of codimension r in $X = \mathbf{P}^n_k$. Show that $\mathrm{cd}(X - Y) \leq r - 1$.

Proof. (a) It suffices to show that if, for some i, $H^i(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$, then $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$. Thus suppose the ith cohomology of all coherent sheaves on X vanishes and let $\mathcal{F}$ be quasi-coherent. Let $(\mathcal{F}_\alpha)$ be the collection of coherent subsheaves of $\mathcal{F}$, ordered by inclusion. Then by (II, Ex. 5.15e) $\varinjlim \mathcal{F}_\alpha = \mathcal{F}$, so by (2.9)

$$H^i(X, \mathcal{F}) = H^i(X, \varinjlim \mathcal{F}_\alpha) = \varinjlim H^i(X, \mathcal{F}_\alpha) = 0.$$

(b) Suppose n is an integer and $H^i(X,\mathcal{F}) = 0$ for all coherent locally free sheaves $\mathcal{F}$ and integers $i > n$. We must show $H^i(X,\mathcal{F}) = 0$ for all coherent $\mathcal{F}$ and all $i > n$, then applying (a) gives the desired result. Since X is quasiprojective there is an open immersion

$$i : X \hookrightarrow Y \subset \mathbf{P}^n_k$$

with Y a closed subscheme of $\mathbf{P}^n_k$ and $i(X)$ open in Y. By (II, Ex. 5.5c) the sheaf $\mathcal{F}$ on X pushes forward to a coherent sheaf on $\mathcal{F}' = i_*\mathcal{F}$ on Y. By (II, 5.18) we may write $\mathcal{F}'$ as a quotient of a locally free coherent sheaf $\mathcal{E}'$ on Y. Letting $\mathcal{R}'$ be the kernel gives an exact sequence

$$0 \to \mathcal{R}' \to \mathcal{E}' \to \mathcal{F}' \to 0$$

with R' coherent (it's the quotient of coherent sheaves). Pulling back via i to X gives an exact sequence

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

of coherent sheaves on X with E locally free. The long exact sequence of cohomology shows that for $i > n$, there is an exact sequence

$$0 = H^i(X,\mathcal{E}) \to H^i(X,\mathcal{F}) \to H^{i+1}(X,\mathcal{R}) \to H^{i+1}(X,\mathcal{E}) = 0.$$

$H^i(X,\mathcal{E}) = H^{i+1}(X,\mathcal{E}) = 0$ because we have assumed that, for $i > n$, cohomology vanishes on locally free coherent sheaves. Thus $H^i(X,\mathcal{F}) \cong H^{i+1}(X,\mathcal{R})$. But if $k = \dim X$, then Grothendieck vanishing (2.7) implies that $H^{k+1}(X,\mathcal{R}) = 0$ whence $H^k(X,\mathcal{F}) = 0$. But then applying the above argument with $\mathcal{F}$ replaced by $\mathcal{R}$ shows that $H^k(X,\mathcal{R}) = 0$ which implies $H^{k-1}(X,\mathcal{F}) = 0$ (so long as $k-1 > n$). Again, apply the entire argument with $\mathcal{F}$ replaced by $\mathcal{R}$ to see that $H^{k-1}(X,\mathcal{R}) = 0$. We can continue this descent and hence show that $H^i(X,\mathcal{F}) = 0$ for all $i > n$.

(c) By (4.5) we can compute cohomology by using the Čech complex resulting from the cover $\mathfrak{U}$ of X by $r+1$ open affines. By definition $\mathcal{C}^p = 0$ for all $p > r$ since there are no intersections of $p+1 \geq r+2$ distinct open sets in our collection of $r+1$ open sets. The Čech complex is

$$\mathcal{C}^0 \to \mathcal{C}^1 \to \cdots \to \mathcal{C}^r \to \mathcal{C}^{r+1} = 0 \to 0 \to 0 \to \cdots.$$

Thus if $\mathcal{F}$ is quasicoherent then $\check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) = 0$ for any $p > r$ which implies that $\mathrm{cd}(X) \leq r$.

(d) I will first present my solution in the special case that X is projective. The more general case when X is quasi-projective is similiar, but more complicated, and will be presented next. Suppose $X \subset \mathbf{P}^n$ is a projective variety of dimension r. We must cover X with $r+1$ open affines. Let U be nonempty open affine subset of X. Since X is irreducible, the irreducible components of $X - U$ all have codimension at least one in X. Now pick a hyperplane H which doesn't completely contain any irreducible component of $X - U$. We can do this by choosing one point P_i in each of the finitely many irreducible components of $X - U$ and choosing a hyperplane which avoids all the P_i. This can be done because the field is infinite (varieties are only defined over algebraically closed fields) so we can always choose a vector not orthogonal to any of a finite set of vectors. Since X is closed in $\mathbf{P}^n$ and $\mathbf{P}^n - H$ is affine, $(\mathbf{P}^n - H) \cap X$ is an open affine subset of X. Because of our choice of H, $U \cup ((\mathbf{P}^n - H) \cap X)$ is only missing codimension two closed subsets of X. Let $H_1 = H$ and choose another hyperplane H_2 so it doesn't completely contain any of the (codimension two) irreducible components of $X - U - (\mathbf{P}^n - H_1)$. Then $(\mathbf{P}^n - H_2) \cap X$ is open affine and $U \cup ((\mathbf{P}^n - H_1) \cap X) \cup ((\mathbf{P}^n - H2) \cap X)$ is only missing codimension three closed subsets of X. Repeating this process a few more times yields hyperplanes $H_1, \cdots, H_r$ so that

$$U, (\mathbf{P}^n - H_1) \cap X, \ldots, (\mathbf{P}^n - H_r) \cap X$$

form an open affine cover of X, as desired.

Now for the quasi-projective case. Suppose $X \subset \mathbf{P}^n$ is quasi-projective. From (I, Ex. 3.5) we know that $\mathbf{P}^n$ minus a hypersurface H is affine. Note that the same proof works even if H is a union of hypersurfaces. We now proceed with the same sort of construction as in the projective case, but we must choose H more cleverly to insure that $(\mathbf{P}^n - H) \cap X$ is affine. Let U be a nonempty affine open subset of X. As before pick a hyperplane which doesn't completey contain any irreducible component of $X - U$. Since X is only quasi-projective we can't conclude that $(\mathbf{P}^n - H) \cap X$ is affine. But we do know that $(\mathbf{P}^n - H) \cap \overline{X}$ is affine. Our strategy is to add some hypersurfaces to H to get a union of hypersurfaces S so that

$$(\mathbf{P}^n - S) \cap \overline{X} = (\mathbf{P}^n - S) \cap X.$$

But, we must be careful to add these hypersurfaces in such a way that $((\mathbf{P}^n - S) \cap X) \cup U$ is missing only codimension two or greater subsets of X. We do this as follows. For each irreducible component Y of $\overline{X} - X$ choose a hypersurface H' which completely contains Y but which does not completely contain any irreducible component of $X - U$. That this can be done is the content of a lemma which will be proved later (just pick a point in each irreducible component and avoid it). Let S by the union of all of the H' along with H. Then $\mathbf{P}^n - S$ is affine and so

$$(\mathbf{P}^n - S) \cap X = (\mathbf{P}^n - S) \cap \overline{X}$$

is affine. Furthermore, S properly intersects all irreducible components of $X - U$, so $((\mathbf{P}^n - S) \cap X) \cup U$ is missing only codimension two or greater subsets of X. Repeating this process as above several times yields the desired result because after each repetition the codimension of the resulting pieces is reduced by 1.

Lemma 1.2. *If Y is a projective variety and $p_1, \dots, p_n$ is a finite collection of points not on Y, then there exists a (possibly reducible) hypersurface H containing Y but not containing any of the p_i.*

By a possibly reducible hypersurface I mean a union of irreducible hypersurfaces, not a hypersurface union higher codimension varieties.

Proof. This is obviously true and I have a proof, but I think there is probably a more algebraic proof. Note that k is infinite since we only talk about varieties over algebraically closed fields. Let $f_1, \cdots, f_m$ be defining equations for Y. Thus Y is the common zero locus of the f_i and not all f_i vanish on any p_i. I claim that we can find a linear combination $\sum a_i f_i$ of the f_i which doesn't vanish on any p_i. Since k is infinite and not all f_i vanish on p_1, we can easily find a_i so that $\sum a_i f_i(p_1) \neq 0$ and all the $a_i \neq 0$. If $\sum a_i f_i(p_2) = 0$ then, once again since k is infinite, we can easily "jiggle" the a_i so that $\sum a_i f_i(p_2) \neq 0$ and $\sum a_i f_i(p_1)$ is still nonzero. Repeating this same argument for each of the finitely many points p_i gives a polynomial $f = \sum a_i f_i$ which doesn't vanish on any p_i. Of course I want to use f to define our hypersurface, but I can't because f might not be homogeneous. Fortunately, this is easily dealt with by suitably multiplying the various f_i by the defining equation of a hyperplane not passing through any p_i, then repeating the above argument. Now let H be the hypersurface defined by $f = \sum a_i f_i$. Then by construction H contains Y and H doesn't contain any p_i. □

(e) Suppose Y is a set-theoretic complete intersection of codimension r in $X = \mathbf{P}^n_k$. Then Y is the intersection of r hypersurfaces, so we can write

$Y = H_1 \cap \cdots \cap H_r$ where each H_i is a hypersurface. By (I, Ex. 3.5) $X - H_i$ is affine for each i, thus

$$X - Y = (X - H_1) \cup \cdots \cup (X - H_r)$$

can be covered by r open affine subsets. By (c) this implies $\operatorname{cd}(X-Y) \le r-1$ which completes the proof. $\square$

Exercise 1.3. **(4.9)** Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y}$ in $\mathbf{P}^4_k$ is not a set-theoretic complete intersection.

Proof. By (Ex. 4.8e) it suffices to show that $H^2(X-Y, \mathcal{O}_{X-Y}) \neq 0$. Suppose Z is a closed subset of X, then by (Ex. 2.3d), for any $i \ge 1$, there is an exact sequence

$$H^i(X, \mathcal{O}_X) \to H^i(X - Z, \mathcal{O}_{X-Z}) \to H^{i+1}_Z(X, \mathcal{O}_X) \to H^{i+1}(X, \mathcal{O}_X).$$

By (3.8), $H^i(X, \mathcal{O}_X) = H^{i+1}(X, \mathcal{O}_X) = 0$ so $H^i(X-Z, \mathcal{O}_{X-Z}) = H^{i+1}_Z(X, \mathcal{O}_X)$. Applying this with $Z = Y$ and $i = 2$ shows that

$$H^2(X - Y, \mathcal{O}_{X-Y}) = H^3_Y(X, \mathcal{O}_X).$$

Thus we just need to show that $H^3_Y(X, \mathcal{O}_X) \neq 0$.

Mayer-Vietoris (Ex. 2.4) yields an exact sequence

$$\begin{gathered} H^3_{Y_1}(X, \mathcal{O}_X) \oplus H^3_{Y_2}(X, \mathcal{O}_X) \to H^3_Y(X, \mathcal{O}_X) \to \\ H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X) \to H^4_{Y_1}(X, \mathcal{O}_X) \oplus H^4_{Y_2}(X, \mathcal{O}_X) \end{gathered}$$

As above, $H^3_{Y_1}(X, \mathcal{O}_X) = H^2(X - Y_1, \mathcal{O}_{X-Y_1})$. But $X - Y_1$ is a set-theoretic complete intersection of codimension 2 so $\operatorname{cd}(X - Y_1) \le 1$, whence $H^2(X - Y_1, \mathcal{O}_{X-Y_1}) = 0$. Similiarly

$$H^2(X - Y_2, \mathcal{O}_{X-Y_2}) = H^3(X - Y_1, \mathcal{O}_{X-Y_1}) = H^3(X - Y_2, \mathcal{O}_{X-Y_2}) = 0.$$

Thus from the above exact sequence we see that $H^3_Y(X, \mathcal{O}_X) = H^4_{Y_1 \cap Y_2}(X, \mathcal{O}_X)$.

Let $P = Y_1 \cap Y_2 = \{(0,0,0,0)\}$. We have reduced to showing that $H^4_P(X, \mathcal{O}_X)$ is nonzero. Since $H^4_P(X, \mathcal{O}_X) = H^3(X - P, \mathcal{O}_{X-P})$ we can do this by a direct computation of $H^3(X - P, \mathcal{O}_{X-P})$ using Čech cohomology.

Cover $X - P$ by the affine open sets $U_i = \{x_i \neq 0\}$. Then the Čech complex is

$$\begin{aligned}
&k[x_1, x_2, x_3, x_4, x_1^{-1}] \oplus \cdots \oplus k[x_1, x_2, x_3, x_4, x_4^{-1}] \xrightarrow{d_0} \\
&k[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}] \oplus \cdots \oplus k[x_1, x_2, x_3, x_4, x_3^{-1}, x_4^{-1}] \xrightarrow{d_1} \\
&k[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}, x_3^{-1}] \oplus \cdots \oplus k[x_1, x_2, x_3, x_4, x_2^{-1}, x_3^{-1}, x_4^{-1}] \xrightarrow{d_2} \\
&k[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}]
\end{aligned}$$

Thus

$$H^3(X - P, \mathcal{O}_{X-P}) = \{x_1^i x_2^j x_3^k x_4^\ell : i, j, k, \ell < 0\} \neq 0.$$

□

Exercise 1.4. **(5.6)** *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}^3_k$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\operatorname{Pic} Q \cong \mathbf{Z} \oplus \mathbf{Z}$, so we can talk about the *type* (a,b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a, b)$. Thus for any $n \in \mathbf{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$.

[**Comment!** In my solution, a subscheme Y of type (a, b) corresponds to the invertible sheaf $\mathcal{O}_Q(-a, -b)$. I think this is reasonable since then $\mathcal{O}_Q(-a, -b) = \mathcal{L}(-Y) = \mathcal{I}_Y$. The correspondence is not clearly stated in the problem, but this choice works.]

(a) Use the special case $(q, 0)$ and $(0, q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q, to show:

1. if $|a - b| \leq 1$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$;
2. if $a, b < 0$, then $H^1(Q, \mathcal{O}_Q(a, b)) = 0$;
3. if $a \leq -2$, then $H^1(Q, \mathcal{O}_Q(a, 0)) \neq 0$).

Solution. First I will prove a big lemma in which I explicitely calculate $H^1(Q, \mathcal{O}_Q(0, -q))$ and some other things which will come in useful later. Next I give an independent computation of the other cohomology groups (1), (2).

Lemma 1.5. *Let $q > 0$, then*

$$\dim_k H^1(Q, \mathcal{O}_Q(-q, 0)) = H^1(Q, \mathcal{O}_Q(0, -q)) = q - 1.$$

Furthermore, we know all terms in the long exact sequence of cohomology associated with the short exact sequence

$$0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Proof. We prove the lemma only for $\mathcal{O}_Q(-q,0)$, since the argument for $\mathcal{O}_Q(0,-q)$ is exactly the same. Suppose Y is the disjoint union of q lines $\mathbf{P}^1$ in Q so $\mathcal{I}_Y = \mathcal{O}_Q(-q,0)$. The sequence

$$0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

is exact. The associated long exact sequence of cohomology is

$$\begin{aligned} 0 \to &\Gamma(Q,\mathcal{O}_Q(-q,0)) \to \Gamma(Q,\mathcal{O}_Q) \to \Gamma(Q,\mathcal{O}_Y) \\ \to &H^1(Q,\mathcal{O}_Q(-q,0)) \to H^1(Q,\mathcal{O}_Q) \to H^1(Q,\mathcal{O}_Y) \\ \to &H^2(Q,\mathcal{O}_Q(-q,0)) \to H^2(Q,\mathcal{O}_Q) \to H^2(Q,\mathcal{O}_Y) \to 0 \end{aligned}$$

We can compute all of the terms in this long exact sequence. For the purposes at hand it suffices to view the summands as k-vector spaces so we systematically do this throughout. Since $\mathcal{O}_Q(-q,0) = \mathcal{I}_Y$ is the ideal sheaf of Y, its global sections must vanish on Y. But $\mathcal{I}_Y$ is a subsheaf of $\mathcal{O}_Q$ whose global sections are the constants. Since the only constant which vanish on Y is 0, $\Gamma(Q,\mathcal{O}_Q(-q,0)) = 0$. By (I, 3.4), $\Gamma(Q,\mathcal{O}_Q) = k$. Since Y is the disjoint union of q copies of $\mathbf{P}^1$ and each copy has global sections k, $\Gamma(Q,\mathcal{O}_Y) = k^{\oplus q}$. Since Q is a complete intersection of dimension 2, (Ex. 5.5 b) implies $H^1(Q,\mathcal{O}_Q) = 0$. Because Y is isomorphic to several copies of $\mathbf{P}^1$, the general result (proved in class, but not in the book) that $H^n_*(\mathcal{O}_{\mathbf{P}^n}) = \{\sum a_I X_I :$ entries in I negative$\}$ implies $H^1(Q,\mathcal{O}_Y) = H^1(Y,\mathcal{O}_Y) = 0$. Since Q is a hypersurface of degree 2 in $\mathbf{P}^3$, (I, Ex. 7.2(c)) implies $p_a(Q) = 0$. Thus by (Ex. 5.5c) we see that $H^2(Q,\mathcal{O}_Q) = 0$. Putting together the above facts and some basic properties of exact sequences show that $H^1(Q,\mathcal{O}_Q(-q,0)) = k^{\oplus(q-1)}$, $H^2(Q,\mathcal{O}_Q(-q,0)) = 0$ and $H^2(Q,\mathcal{O}_Y) = 0$. Our long exact sequence is now

$$\begin{aligned} 0 \to &\Gamma(Q,\mathcal{O}_Q(-q,0)) = 0 \to \Gamma(Q,\mathcal{O}_Q) = k \to \Gamma(Q,\mathcal{O}_Y) = k^{\oplus q} \\ \to &H^1(Q,\mathcal{O}_Q(-q,0)) = k^{\oplus(q-1)} \to H^1(Q,\mathcal{O}_Q) = 0 \to H^1(Q,\mathcal{O}_Y) = 0 \\ \to &H^2(Q,\mathcal{O}_Q(-q,0)) = 0 \to H^2(Q,\mathcal{O}_Q) = 0 \to H^2(Q,\mathcal{O}_Y) = 0 \to 0 \end{aligned}$$

□

Number (3) now follows immediately from the lemma because

$$H^1(Q, \mathcal{O}_Q(a,0)) = k^{\oplus(-a-1)} \neq 0$$

for $a \leq -2$.

Now we compute (1) and (2). Let a be an arbitrary integer. First we show that $\mathcal{O}_Q(a,a) = 0$. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-2) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_Q \to 0$$

where the first map is multiplication by $xy - zw$. Twisting by a gives an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-2+a) \to \mathcal{O}_{\mathbf{P}^3}(a) \to \mathcal{O}_Q(a) \to 0.$$

The long exact sequence of cohomology yields an exact sequence

$$\cdots \to H^1(\mathcal{O}_{\mathbf{P}^3}(a)) \to H^1(\mathcal{O}_Q(a)) \to H^2(\mathcal{O}_{\mathbf{P}^3}(-2+a)) \to \cdots$$

But from the explicit computations of projective space (5.1) it follows that $H^1(\mathcal{O}_{\mathbf{P}^3}(a)) = 0$ and $H^2(\mathcal{O}_{\mathbf{P}^3}(-2+a)) = 0$ from which we conclude that $H^1(\mathcal{O}_Q(a)) = 0$.

Next we show that $\mathcal{O}_Q(a-1,a) = 0$. Let Y be a single copy of $\mathbf{P}^1$ sitting in Q so that Y has type $(1,0)$. Then we have an exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

But $\mathcal{I}_Y = \mathcal{O}_Q(-1,0)$ so this becomes

$$0 \to \mathcal{O}_Q(-1,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Now twisting by a yields the exact sequence

$$0 \to \mathcal{O}_Q(a-1,a) \to \mathcal{O}_Q(a) \to \mathcal{O}_Y(a) \to 0.$$

The long exact sequence of cohomology gives an exact sequence

$$\cdots \to \Gamma(\mathcal{O}_Q(a)) \to \Gamma(\mathcal{O}_Y(a)) \to H^1(\mathcal{O}_Q(a-1,a)) \to H^1(\mathcal{O}_Q(a)) \to \cdots$$

We just showed that $H^1(\mathcal{O}_Q(a)) = 0$, so to see that $H^1(\mathcal{O}_Q(a-1,a)) = 0$ it suffices to note that the map $\Gamma(\mathcal{O}_Q(a)) \to \Gamma(\mathcal{O}_Y(a))$ is surjective. This can be seen by writing $Q = \mathrm{Proj}(k[x,y,z,w]/(xy-zw))$ and (w.l.o.g.)

$Y = \mathrm{Proj}(k[x,y,z,w]/(xy-zw,x,z))$ and noting that the degree a part of $k[x,y,z,w]/(xy-zw)$ surjects onto the degree a part of $k[x,y,z,w]/(xy-zw,x,z)$. Thus $H^1(\mathcal{O}_Q(a-1,a)) = 0$ and exactly the same argument shows $H^1(\mathcal{O}_Q(a,a-1)) = 0$. This gives (1).

For (2) it suffices to show that for $a > 0$,

$$H^1(\mathcal{O}_Q(-a,-a-n)) = H^1(\mathcal{O}_Q(-a-n,-a)) = 0$$

for all $n > 0$. Thus let $n > 0$ and suppose Y is a disjoint union of n copies of $\mathbf{P}^1$ in such a way that $\mathcal{I}_Y = \mathcal{O}_Q(0,-n)$. Then we have an exact sequence

$$0 \to \mathcal{O}_Q(0,-n) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Twisting by $-a$ yields the exact sequence

$$0 \to \mathcal{O}_Q(-a,-a-n) \to \mathcal{O}_Q(-a) \to \mathcal{O}_Y(-a) \to 0.$$

The long exact sequence of cohomology then gives an exact sequence

$$\cdots \to \Gamma(\mathcal{O}_Y(-a)) \to H^1(\mathcal{O}_Q(-a,-a-n)) \to H^1(\mathcal{O}_Q(-a)) \to \cdots$$

As everyone knows, since Y is just several copies of $\mathbf{P}^1$ and $-a < 0$, $\Gamma(\mathcal{O}_Y(-a)) = 0$. Because of our computations above, $H^1(\mathcal{O}_Q(-a)) = 0$. Thus $H^1(\mathcal{O}_Q(-a,-a-n)) = 0$, as desired. Showing that $H^1(\mathcal{O}_Q(-a-n,-a)) = 0$ is exactly the same.

(b) Now use these results to show:

1. If Y is a locally principal closed subscheme of type (a,b) with $a,b > 0$, then Y is connected.

   Proof. Computing the long exact sequence associated to the short exact sequence

   $$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

   gives the exact sequence

   $$0 \to \Gamma(Q,\mathcal{I}_Y) \to \Gamma(Q,\mathcal{O}_Q) \to \Gamma(Q,\mathcal{O}_Y) \to H^1(Q,\mathcal{I}_Y) \to \cdots$$

   But, $\Gamma(\mathcal{I}_Y) = 0$, $\Gamma(Q,\mathcal{O}_Q) = k$, and by (a)2 above $H^1(Q,\mathcal{I}_Y) = H^1(Q,\mathcal{O}_Q(-a,-b)) = 0$. Thus we have an exact sequence

   $$0 \to 0 \to k \to \Gamma(\mathcal{O}_Y) \to 0 \to \cdots$$

   from which we conclude that $\Gamma(\mathcal{O}_Y) = k$ which implies Y is connected. $\square$

2. now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a, b). Use (II, 7.6.2) and (II, 8.18).

   Proof. Given (a, b), (II, 7.6.2) gives a closed immersion

   $$Q = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^a \times \mathbf{P}^b \to \mathbf{P}^n$$

   which corresponds to the invertible sheaf $\mathcal{O}_Q(-a, -b)$ of type (a, b). By Bertini's theorem (II, 8.18) there is a hyperplane H in $\mathbf{P}^n$ such that the hyperplane section of the (a, b) embedding of Q in $\mathbf{P}^n$ is nonsingular. Pull this hyperplane section back to a nonsingular curve Y of type (a, b) on Q in $\mathbf{P}^3$. By the previous problem, Y is connected. Since Y comes from a hyperplane section this implies Y is irreducible (see the remark in the statement of Bertini's theorem). $\square$

3. an irreducible nonsingular curve Y of type (a, b), $a, b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a-b| \leq 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type $(1, 3)$ which is just the rational quartic curve (I, Ex. 3.18).

   Proof. Let Y be an irreducible nonsingular curve of type (a, b). The criterion we apply comes from (II, Ex 5.14d) which asserts that the maps

   $$\Gamma(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$$

   are surjective for all $n \geq 0$ if and only if Y is projectively normal. To determine when this occurs we have to replace $\Gamma(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n))$ with $\Gamma(Q, \mathcal{O}_Q(n))$. It is easy to see that the above criterion implies we can make this replacement if Q is projectively normal. Since $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$ is locally isomorphic to $\mathbf{A}^1 \times \mathbf{A}^1 \cong \mathbf{A}^2$ which is normal, we see that Q is normal. Then since Q is a complete intersection which is normal, (II, 8.4b) implies Q is projectively normal.

   Consider the exact sequence

   $$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y.$$

Twisting by n gives an exact sequence

$$0 \to \mathcal{I}_Y(n) \to \mathcal{O}_Q(n) \to \mathcal{O}_Y(n).$$

Taking cohomology yields the exact sequence

$$\cdots \to \Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Q, \mathcal{O}_Y(n)) \to H^1(Q, \mathcal{I}_Y(n)) \to \cdots$$

Thus Y is projectively normal precisely if $H^1(Q, \mathcal{I}_Y(n)) = 0$ for all $n \geq 0$. When can this happen? We apply our computations from part (a). Since $\mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$,

$$\mathcal{I}_Y(n) = \mathcal{O}_Q(-a, -b)(n) = \mathcal{O}_Q(-a, -b) \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(n, n) = \mathcal{O}_Q(n-a, n-b)$$

If $|a - b| \leq 1$ then $|(n-a) - (n-b)| \leq 1$ for all n so

$$H^1(Q, \mathcal{O}_Q(-a, -b)(n)) = 0$$

for all n which implies Y is projectively normal. On the other hand, if $|a - b| > 1$ let n be the minimum of a and b, without loss assume b is the minimum, so $n = b$. Then from (a) we see that

$$\mathcal{O}_Q(-a, -b)(n) = \mathcal{O}_Q(-a, -b)(b) = \mathcal{O}_Q(-a + b, 0) \neq 0$$

since $-a + b \leq -2$. $\square$

(c) If Y is a locally principal subscheme of type (a, b) in Q, show that $p_a(Y) = ab - a - b + 1$. [Hint: Calculate the Hilbert polynomials of suitable sheaves, and again use the special case (q,0) which is a disjoint union of q copies of $\mathbf{P}^1$.]

Proof. The sequence

$$0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

is exact so

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Q) - \chi(\mathcal{O}_Q(-a, -b)) = 1 - \chi(\mathcal{O}_Q(-a, -b)).$$

Thus

$$p_a(Y) = 1 - \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Q(-a, -b)).$$

The problem is thus reduced to computing $\chi(\mathcal{O}_Q(-a,-b))$.

Assume first that $a, b < 0$. To compute $\chi(\mathcal{O}_Q(-a,-b))$ assume $Y = Y_1 \cup Y_2$ where $\mathcal{I}_{Y_1} = \mathcal{O}_Q(-a, 0)$ and $\mathcal{I}_{Y_2} = \mathcal{O}_Q(0, -b)$. Thus we could take Y_1 to be a copies of $\mathbf{P}^1$ in one family of lines and Y_2 to be b copies of $\mathbf{P}^1$ in the other family. Tensoring the exact sequence

$$0 \to \mathcal{I}_{Y_1} \to \mathcal{O}_Q \to \mathcal{O}_{Y_1} \to 0$$

by the flat module $\mathcal{I}_{Y_2}$ yields an exact sequence

$$0 \to \mathcal{I}_{Y_1} \otimes \mathcal{I}_{Y_2} \to \mathcal{I}_{Y_2} \to \mathcal{O}_{Y_1} \otimes \mathcal{I}_{Y_2}$$

[Note: I use the fact that $\mathcal{I}_{Y_2}$ is flat. This follows from a proposition in section 9 which we haven't yet reached, but I'm going to use it anyways. Since Y_2 is locally principal, $\mathcal{I}_{Y_2}$ is generated locally by a single element and since Q is a variety it is integral. Thus $\mathcal{I}_{Y_2}$ is locally free so by (9.2) $\mathcal{I}_{Y_2}$ is flat.] This exact sequence can also be written as

$$0 \to \mathcal{O}_Q(-a,-b) \to \mathcal{O}_Q(0,-b) \to \mathcal{O}_Y \otimes \mathcal{O}_Q(0,-b) \to 0.$$

The associated long exact sequence of cohomology is

$$\begin{aligned}
0 \to &\Gamma(Q, \mathcal{O}_Q(-a,-b)) \to \Gamma(Q, \mathcal{O}_Q(0,-b)) \to \Gamma(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) \\
\to &H^1(Q, \mathcal{O}_Q(-a,-b)) \to H^1(Q, \mathcal{O}_Q(0,-b)) \to H^1(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) \\
\to &H^2(Q, \mathcal{O}_Q(-a,-b)) \to H^2(Q, \mathcal{O}_Q(0,-b)) \to H^2(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) \to 0
\end{aligned}$$

The first three groups of global sections are 0. Since $a, b < 0$, (a) implies $H^1(Q, \mathcal{O}_Q(-a,-b)) = 0$. From the lemma we know that $H^1(Q, \mathcal{O}_Q(0,-b)) = k^{\oplus(b-1)}$. Also by the lemma we know that $H^2(Q, \mathcal{O}_Q(0,-b)) = 0$. Since $\mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)$ is isomorphic to the ideal sheaf of $b-1$ points in each line of Y_1, a similiar proof as that used in the lemma shows that

$$H^1(Q, \mathcal{O}_Y \otimes \mathcal{O}_Q(0,-b)) = k^{\oplus a(b-1)}.$$

Plugging all of this information back in yields the exact sequence

$$\begin{aligned}
0 \to &\Gamma(Q, \mathcal{O}_Q(-a,-b)) = 0 \to \Gamma(Q, \mathcal{O}_Q(0,-b)) = 0 \to \Gamma(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) = 0 \\
\to &H^1(Q, \mathcal{O}_Q(-a,-b)) = 0 \to H^1(Q, \mathcal{O}_Q(0,-b)) = k^{\oplus(b-1)} \\
&\qquad \to H^1(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) = k^{\oplus a(b-1)} \\
\to &H^2(Q, \mathcal{O}_Q(-a,-b)) \to H^2(Q, \mathcal{O}_Q(0,-b)) = 0 \\
&\qquad \to H^2(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) = 0 \to 0
\end{aligned}$$

From this we conclude that

$$\chi(\mathcal{O}_Q(-a,-b)) = 0+0+h^2(Q,\mathcal{O}_Q(-a,-b)) = a(b-1)-(b-1) = ab-a-b+1$$

which is the desired result.

Now we deal with the remaining case, when Y is a disjoint copies of $\mathbf{P}^1$. We have

$$p_a(Y) = 1-\chi(\mathcal{O}_Y) = 1-\chi(\mathcal{O}_{\mathbf{P}^1}^{\oplus a}) = 1-a\chi(\mathcal{O}_{\mathbf{P}^1}) = 1-a$$

which completes the proof. $\square$