Homework 2, MAT256B
II.8.4, III.6.8, III.7.1, III.7.3

William A. Stein

April 2, 1996

1 Exercise II.8.4

Complete Intersections in $\mathbf{P}^n$. *A closed subscheme* Y *of* $\mathbf{P}^n_k$ *is called a* (strict, global) complete intersection *if the homogenous ideal* I *of* Y *in* $S = k[x_0, \ldots, x_n]$ *can be generated by* r *elements where* $r = \operatorname{codim}(Y, \mathbf{P}^n)$.

(a) Let Y *be a closed subscheme of codimension* r *in* $\mathbf{P}^n$. *Then* Y *is a complete intersection iff there are hypersurfaces (i.e., locally principal subschemes of codimension* 1*)* $H_1, \ldots, H_r$, *such that* $Y = H_1 \cap \cdots \cap H_r$ as schemes, *i.e.,* $\mathcal{I}_Y = \mathcal{I}_{H_1} + \cdots + \mathcal{I}_{H_r}$.

($\Rightarrow$) By (II, Ex 5.14) I is defined to be $\Gamma_*(\mathcal{I}_Y)$. By (II, 5.15), $\tilde{I} \cong \mathcal{I}_Y$. Write $I = (f_1, \ldots, f_r)$, then since localization commutes with taking sums,

$$\mathcal{I}_Y = (f_1, \ldots, f_r)\tilde{} = ((f_1) + \cdots + (f_r))\tilde{} = (f_1)\tilde{} + \cdots + (f_r)\tilde{}.$$

Let H_i be the locally principal closed subscheme of codimension 1 determined by the ideal sheaf $(f_i)\tilde{}$. Then Y is the intersection of the H_i.

($\Leftarrow$) Someone suggested I should apply unmixedness and primary decomposition to some ideal somewhere and use the fact that a saturated ideal doesn't have primary components corresponding to the irrelevant ideal or something like that. NOT DONE.

(b) If Y *is a complete intersection of dimension* ≥ 1 *in* $\mathbf{P}^n$, *and if* Y *is normal, then* Y *is projectively normal (Ex. 5.14).*

Let Z be the cone over Y, then $A(Z) = S/I(Y)$. By (I, Ex. 3.17d), $A(Z)$ is integrally closed iff Z is normal. By definition $A(Z)$ is integrally closed iff Y is projectively normal. Thus we must show that Z is normal.

Since Y is a complete intersection, $I(Y) = (f_1, \dots, f_r)$ so Z is a complete intersection subscheme of $\mathbf{A}^{n+1}$. By (II, 8.23) Z is normal iff Z is regular in codimension 1. Also by (II, 8.23) Y is regular in codimension 1 because we have assumed Y is normal. But Y regular in codimension 1 implies Z regular in codimension 1. [We used this last semester in (II, Ex. 6.3d). Intuitively, the only singularity in Z not in Y is the cone point which has codimension > 1. This is because Z is locally $U_i \times \mathbf{A}^1$.]

(c) With the same hypothesis as in (b), conclude that for all $\ell \geq 0$, the natural map $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell)) \to \Gamma(Y, \mathcal{O}_Y(\ell))$ is surjective. In particular, taking $\ell = 0$, show that Y is connected.

That the map $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell)) \to \Gamma(Y, \mathcal{O}_Y(\ell))$ is surjective is just the statement of (II, Ex. 5.14d). When $\ell = 0$ this says that $k = \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell))$ surjects onto $\Gamma(Y, \mathcal{O}_Y(\ell))$ so $\dim \Gamma(Y, \mathcal{O}_Y) \leq 1$ and hence Y is connected. [If Y were not connected then $\Gamma(Y, \mathcal{O}_Y) = k \oplus \cdots \oplus k$ where the number of direct summands equals the number of components of Y.]

(d) Now suppose given integers $d_1, \dots, d_r \geq 1$, with $r < n$. Use Bertini's theorem (8.18) to show that there exists nonsingular hypersurfaces $H_1, \dots, H_r$ in $\mathbf{P}^n$, with $\deg H_i = d_i$, such that the scheme $Y = H_1 \cap \cdots \cap H_r$ is irreducible and nonsingular in codimension r in $\mathbf{P}^n$.

[To apply Bertini's theorem we must assume k is algebraically closed. I'm going to make this assumption now. Maybe there is a way around this?]

Let $\mathbf{P}^n_k \hookrightarrow \mathbf{P}^{d_1\text{-uple}}_k$ be the d_1-uple embedding of $\mathbf{P}^n_k$. Use Bertini's theorem to choose a hyperplane in $\mathbf{P}^{d_1\text{-uple}}_k$ which has nonsingular intersection with the image of $\mathbf{P}^n_k$. It pulls back to a degree d_1 nonsingular hypersurface H_1 in $\mathbf{P}^n_k$. If $r > 1$ consider the d_2-uple embedding $\mathbf{P}^n_k \hookrightarrow \mathbf{P}^{d_2\text{-uple}}_k$. The image of H_1 is a nonsingular variety in $\mathbf{P}^{d_2\text{-}uple}_k$ of dimension ≥ 2. By Bertinni's theorem there is a hyperplane in $\mathbf{P}^{d_2-uple}_k$ whose intersection with the image of H_1 is nonsingular and of dimension one less than H_1. Pulling back we obtain a hypersurface H_2 such that $H_1 \cap H_2$ is nonsingular and H_2 has degree d_2. Continuing inductively in this way and noting that $\dim H_1 \cap \cdots \cap H_{r-1} \geq 2$ (since $r < n$) completes the proof.

(e) If Y is a nonsingular complete intersection as in (d) show that $\omega_Y \cong \mathcal{O}_Y(\sum d_i - n - 1)$.

By (III, 8.20) $\omega_{H_1} \cong \omega_{\mathbf{P}^n} \otimes \mathcal{L}(H_1) \otimes \mathcal{O}_{H_1}$. By the explicit computation of $Cl\mathbf{P}^n$ (II, 6.17) we know that $\mathcal{L}(H_1) \cong \mathcal{O}_{\mathbf{P}^n}(d_1)$. Thus

$$\omega_{H_1} \cong \mathcal{O}_{\mathbf{P}^n}(-n-1) \otimes \mathcal{O}_{\mathbf{P}^n}(d_1) \otimes \mathcal{O}_{H_1} \cong \mathcal{O}_{H_1}(d_1 - n - 1).$$

By (8.20) we have that

$$\omega_{H_1\cap H_2} \cong \omega_{H_1} \otimes \mathcal{L}(H_2.H_1) \otimes \mathcal{O}_{H_1\cap H_2}.$$

We know that $H_2 \sim d_2\mathbf{P}^{n-1}$ (linear equivalence) so by (II, 6.2b) this implies $H_2.H_1 \sim d_2\mathbf{P}^{n-1}.H_1$. But $d_2\mathbf{P}^{n-1}.H_1$ corresponds to the invertible sheaf (see (II, Ex 6.8c) $\mathcal{O}_{H_1}(d_2)$. Thus

$$\omega_{H_1\cap H_2} \cong \mathcal{O}_{H_1}(d_1-n-1) \otimes \mathcal{O}_{H_1}(d_2) \otimes \mathcal{O}_{H_1\cap H_2} \cong \mathcal{O}_{H_1\cap H_2}(d_1+d_2-n-1).$$

Repeating this argument inductively yields the desired isomorphism.

(f) If Y is a nonsingular hypersurface of degree d in $\mathbf{P}^n$, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (I, Ex. 7.2).

By definition $p_g(Y) = \dim_k \Gamma(Y, \omega_Y)$. By (e), $\omega_Y \cong \mathcal{O}(d-n-1)$ and by (c) the natural map

$$\Gamma(X, \mathcal{O}_X(d-n-1)) \to \Gamma(Y, \mathcal{O}_Y(d-n-1))$$

is surjective. We show that it is also injective. By (III, 5.5a) an element $f \in \Gamma(X, \mathcal{O}_X(d-n-1))$ can be represented as a homogeneous polynomial of degree $d-n-1$. Now f maps to 0 in $\Gamma(Y, \mathcal{O}_Y(d-n-1))$ iff f vanishes on Y, that is to say, $Y \subset Z(f)$. But $\deg Y = d > d-n-1 = \deg f$ so Y can not be contained in the hypersurface $Z(f)$ unless $f = 0$. [Proof: $Y = Z(g) \subset Z(f)$ implies $(f) \subset (g)$ so f is a multiple of g, but g has degree strictly greater than f so must be 0.] Thus

$$\dim_k \Gamma(Y, \mathcal{O}_Y(d-n-1)) = \dim_k \Gamma(X, \mathcal{O}_X(d-n-1)) = \binom{d-1}{n}$$

since the number of monomoials in $k[x_0, \ldots, x_n]$ of degree $d-n-1$ is $\binom{d-1}{n}$ as desired.

(g) If Y is a nonsingular curve in $\mathbf{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d, e, then $p_g(Y) = \frac{1}{2}de(d+e-4)+1$. Again the geometric genus is the same as the arithmetic genus (I, Ex. 7.2).

Let H be the hypersurface of degree d. There is an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-d) \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_H \to 0.$$

Twisting by a and computing dimensions we see that

$$\dim \mathcal{O}_H(a) = \dim \mathcal{O}_{\mathbf{P}^n}(a) - \dim \mathcal{O}_{\mathbf{P}^n}(a-d) = \binom{3+a}{3} - \binom{3+a-d}{3}.$$

Using reasoning like that in (e) we obtain an exact sequence

$$0 \to \mathcal{O}_H(-e) \to \mathcal{O}_H \to \mathcal{O}_Y \to 0.$$

Twisting by $e+d-4$ yields the exact sequence

$$0 \to \mathcal{O}_H(d-4) \to \mathcal{O}_H(e+d-4) \to \mathcal{O}_Y(e+d-4) \to 0.$$

Applying the above explicit computation of $\dim \mathcal{O}_H(a)$ we see that

$$\dim \mathcal{O}_Y(e+d-4) = \binom{e+d-1}{3} - \binom{e-1}{3} - \binom{d-1}{3} + chose{-}13.$$

After some algebra the latter expression becomes $\frac{1}{2}ed(e+d-4)+1$, as desired.

[Comment 1: We could have also solved (f) using this method.]

[Comment 2: Serre duality gives another solution. By (III, 7.12.4)

$$p_g(Y) = \dim H^0(Y, \omega_Y) = \dim H^1(Y, \mathcal{O}_Y) = p_a(Y).$$

But by (I, Ex. 7.2d) $p_a(Y) = \frac{1}{2}de(d+e-4)+1$.]

2 Exercise III.6.8

Prove the following theorem of Kleiman: if X is a noetherian, integral, seperated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

(a) First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathcal{L})$ and various invertible sheaves $\mathcal{L}$ on X, form a base for the topology of X.

Let $x \in U \subset X$ with U open.

Case 1. $W = X - U$ is irreducible. Since $x \notin W$, $\mathcal{O}_x \not\subset \mathcal{O}_W$. [This assertion is a matter of some difficulty among the others working on this problem. It is not hard to see when X is a variety in the classical sense. But in the more general situation it isn't at all clear and may use the hypothesis that X is seperated in an essential way. For example, the affine line with a doubled origin has two different local rings which are equal. I'm not sure how to resolve this but there was some talk of using the valuative criterion for seperatedness. PUT CORRECT SOLUTION HERE AFTERWARDS.] Thus let $h \in K$ be a rational function such that $h \notin \mathcal{O}_W$ but $h \in \mathcal{O}_x$. Let $(h) = D_1 - D_2$ with D_1=zeros of h and D_2=poles of h. Since $h \in \mathcal{O}_x$, we have $x \notin \mathrm{Var}(D_2)$=the underlying scheme of the effective divisor D_2.

(This is because h can't have a pole at x.) Furthermore $y \in W$ implies $\mathcal{O}_y \subset \mathcal{O}_W$ so $h \notin \mathcal{O}_y$ thus $y \in \mathrm{Var}(D_2)$ (this is because X is factorial so $\mathcal{O}_y$ is integrally closed so $v_y(h) < 0$ iff $h \notin \mathcal{O}_y$.) Thus $W \subset \mathrm{Var}(D_2)$. Since X is factorial and D_2 is effective (II, 6.11) implies D_2 corresponds to an effective Cartier divisor and hence there exists an open cover $\mathfrak{U} = (U_i)$ of X and rational functions $h_i \in K$ such that $h_i|U_i \in \mathcal{O}_{U_i}$ and $(h_i) = D_2$ on U_i. Since $\frac{h_i}{h_j} \in \mathcal{O}_x(U_i \cap U_j)^*$ and X is normal, $(\frac{h_i}{h_j}) = 0$. Let $\mathcal{L}$ be the locally free invertible sheaf represented by the Cartier divisor (U_i, h_i) (so $\mathcal{L}$ is locally generated by $1/h_i$ on U_i), and let $u_i : \mathcal{L}|U_i \to \mathcal{O}_x|U_i$ be the isomorphism given by multiplication by h_i. Define $s(y) = u_i^{-1}(h_i(y))$ for $y \in U_i$. By this we mean u_i^{-1} of the map $y \mapsto h_i(y)$, i.e., s is the glueing of the inverse images of the $h_i \in \mathcal{O}_x(u_i)$. Thus s is a section of $\mathcal{L}$ such that $X_s \cap U_i = U_i - \mathrm{Var}(D_2)$. Thus $X_s = X - \mathrm{Var}(D_2) \subset U$.

Case 2. $W = X - U$ is reducible. Using the fact that X is noetherian write $W = Z_1 \cup \cdots \cup Z_n$. From case 1 we know that there exists invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_n$ and sections $s_i \in \Gamma(X, \mathcal{L}_i)$, $i = 1, \ldots, n$ such that $x \in X_{s_i} \subset X - Z_i$. Let $s = s_1 \otimes \cdots \otimes s_n \in \Gamma(X, \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. Then $X_s = \cap_{i=1}^n X_{s_i}$ hence $x \in X_s \subset U$.

[[This proof was copied from Borelli's paper with little modification. One danger is that the corresponding theorem in Borelli's paper assumes X to be a factorial *variety*, not a more general scheme as above. Part (b) below was not in Borelli.]]

(b) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\oplus \mathcal{L}_i^{n_i}$ for various invertible sheaves $\mathcal{L}_i$ and various integers n_i.

Let $\mathcal{F}$ be a coherent sheaf on X. Let U be an open set on which $\mathcal{F}_{|U} \cong \tilde{M}$. Suppose $X_f \subset U$ where f is a global section of some invertible sheaf $\mathcal{L}$. Our strategy is to construct an appropriate map $\oplus \mathcal{L}_i^{n_i} \to \mathcal{F}$ which is surjective when restricted to X_f, then use the fact that X is noetherian and that the X_f form a basis for the topology on X to cover X which such X_f and then take the sum of all the resulting maps.

Let $m_1, \ldots, m_r$ generate M. Let $t_1, \ldots, t_r$ be the restrictions of the m_i to X_f. By (II, 5.14b) there exists n so that

$$t_1 f^n, \ldots, t_r f^n \in \Gamma(X_{f^n}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$$

extend to global sections $s_1, \ldots, s_n$ of $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. Define a map

$$\oplus_{i=1}^n \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

by sending $(0,\dots,0,1,0,\dots,0)$ (1 in the ith position only) to s_i. Then tensoring with $(\mathcal{L}^{\otimes n})^{-1} = (\mathcal{L}^{-1})^{\otimes n}$ we obtain a map

$$\Theta : \oplus_{i=1}^{n} (\mathcal{L}^{-1})^{\otimes n} \to \mathcal{F}.$$

The map Θ is surjective when restricted to X_f. To see this let p be a point of X_f. The stalk of $\mathcal{F}$ at p is generated by the stalks of $m_1,\dots,m_n$ at p. Since the t_i are all in the image of the map Θ and the stalks of the t_i at p are the same as the stalks of the m_i at p it follows that the stalks of the m_i are all in the image under Θ of the stalk of $\oplus_{i=1}^{n}(\mathcal{L}^{-1})^{\otimes n}$ at p.

Take the direct sum of all such maps over a suitable open cover (U) of X and suitable open covers (X_f) of each U. Since X is noetherian we can arrange it so this sum is finite.

3 Exercise III.7.1

Let X be an integral projective scheme of dimension ≥ 1 over a field k, and let $\mathcal{L}$ be an ample invertible sheaf X. Then $H^0(X, \mathcal{L}^{-1}) = 0$.

Lemma 3.1. *If $\mathcal{M} \neq \mathcal{O}_X$ is an invertible sheaf which is generated by its global sections then $H^0(X, \mathcal{M}^{-1}) = 0$.*

Proof. By the proof of (II 6.12) $\mathcal{M}^{-1} = \mathcal{M}^{\vee} = \mathcal{H}om(\mathcal{M}, \mathcal{O}_X)$. Thus we must show that $\Gamma(X, \mathcal{H}om(\mathcal{M}, \mathcal{O}_X)) = 0$, i.e., that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) = 0$. Since $\mathcal{M}$ is generated by global sections (m_i) to give a morphism $f : \mathcal{M} \to \mathcal{O}_X$ is the same as to give the images $\alpha_i = f(m_i) \in \Gamma(X, \mathcal{O}_X)$ of the m_i. Since X is integral and projective $\Gamma(X, \mathcal{O}_X) = k$ so the α_i all lie in k. Thus if f is nonzero then some $\alpha_i \neq 0$ so $\frac{1}{\alpha_i} m_i \mapsto 1$. Let $t = \frac{1}{\alpha_i} m_i \in \Gamma(X, \mathcal{M})$. Let P be any point of X. The map $f_P : \mathcal{M}_P \to \mathcal{O}_{X,p}$ sends t_p to 1 so it is surjective being a map of free $\mathcal{O}_{X,p}$-modules (and since $\mathcal{O}_{X,p}$ is generated by 1 as an $\mathcal{O}_{X,p}$-module). On the other hand $\mathcal{M}_p$ is free of rank 1 over the integral domain $\mathcal{O}_{X,p}$ so f must be injective. Indeed, if $\mathcal{M}_p \cong \mathcal{O}_{X,p} \cdot g$ for some g and $ag \mapsto 0$ then $af(g) = 0$ so since $\mathcal{O}_{X,p}$ is a domain, $a = 0$ or $f(g) = 0$. But $f(g) \neq 0$ since f is surjective so $a = 0$ and so $ag = 0$ whence f is injective. Therefore f is an isomorphism since it is an isomorphism on stalks. Thus $\mathcal{M} \cong \mathcal{O}_X$ contrary to our assumption that $\mathcal{M} \ncong \mathcal{O}_X$ so there can be no nonzero f in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) = \Gamma(X, \mathcal{M}^{-1})$, as desired. $\square$

Suppose that $\mathcal{L}$ is ample. If $\mathcal{L} = \mathcal{O}_X$ then $\mathcal{L}$ can not be ample, for if $\mathcal{O}_X$ is ample then since $\mathcal{O}_X^{\otimes n} = \mathcal{O}_X$ for any $n \geq 1$ it follows by (II.7.5) that $\mathcal{O}_X$ is very ample. This means that there is an immersion $i : X \hookrightarrow \mathbf{P}_k^n$ where $n = \dim \Gamma(X, \mathcal{O}_X) - 1 = 0$ which is impossible because X has dimension at least 1.

Thus we may assume $\mathcal{L} \not\cong \mathcal{O}_X$ and apply the above lemma. There is an n so that $\mathcal{L}^{\otimes n}$ is generated by its global sections. By the above lemma $H^0(X, (\mathcal{L}^{\otimes n})^\vee) = 0$. Since the collection of invertible sheaves forms a group and $\vee$ is the inverse operation it follows trivially that $(\mathcal{L}^{\otimes n})^\vee \cong (\mathcal{L}^\vee)^{\otimes n}$ and hence $\Gamma(X, (\mathcal{L}^\vee)^{\otimes n}) = 0$. Suppose $\mathcal{L}^\vee$ has a nonzero global section s. Let $p \in X$ be a point so that $s_p \neq 0$. It follows that $s \otimes \cdots \otimes s \neq 0$ in $(\mathcal{L}_p^\vee)^{\otimes n}$. Thus s defines a nonzero global section $s \otimes \cdots \otimes s$ of $(\mathcal{L}^\vee)^{\otimes n}$. [This last statement is a bit subtle because the tensor product is the sheaf associated to a certain presheaf so we don't know, *a priori*, that $s \otimes \cdots \otimes s$ maps to something nonzero under the θ of (II, Defn 1.2). But if $\theta(s \otimes \cdots \otimes s) = 0$ then $0 = \theta(s \otimes \cdots \otimes s)_p = (s \otimes \cdots \otimes s)_p$ so θ is not injective on stalks contradicting the comment after (II, Defn 1.2).] Thus if $H^0(X, \mathcal{L}^\vee) \neq 0$ then $H^0(X, (\mathcal{L}^{\otimes n})^\vee) \neq 0$, a contradiction. It follows that $H^0(X, \mathcal{L}^\vee) = 0$, as desired.

4 Exercise III.7.3

Let $X = \mathbf{P}_k^n$. Show that $H^q(X, \Omega_X^p) = 0$ for $p \neq q$, k for $p = q$, $0 \leq p, q \leq n$.

Our strategy is to use the exact sequence

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0$$

of (II, 8.13) along with (II, Ex 5.16 d) to reduce the computation of the cohomology of Ω_X^p to the computation of the cohomology of $\Lambda^p \mathcal{O}_X(-1)^{\oplus n+1}$. We then show inductively that the cohomology of $\Lambda^p \mathcal{O}_X(-1)^{\oplus n+1}$ vanishes for $p \geq 1$ thus completing the proof.

We compute the cohomology of Ω^r inductively on r.

Step 1, $r = 0$. Suppose $r = 0$ so $\Omega^r = \mathcal{O}_X$. Then by (III, 5.5) $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. [Part (a) of (III, 5.5) gives $H^0(X, \mathcal{O}_X) = k$, part (b) gives $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$ and part (d) gives $H^n(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X(-n-1))^\vee = 0$.]

Step 2 Show that

$$H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}) = 0$$

for $r \geq 1$.

[Matt Baker pointed out to me that

$$\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} \cong \mathcal{O}_X(-1)^{\oplus \binom{n+1}{r}}.$$

This is reasonable since it is true on stalks. It immediately implies the vanishing of the cohomology groups. My original more complicated proof of step 2 is included next anyways.]

Step 2a, $r = 1$. We treat $r = 1$ as a special case. We must show that $H^i(X, \mathcal{O}_X(-1)^{\oplus n+1} = 0$ or equivalently that $H^i(X, \mathcal{O}_X(-1)) = 0$. This is immediate from the explicit computations of (III, 5.5). The argument proceeds exactly as in step 1.

Step 2b, $r \geq 2$. We now assume $r \geq 2$ and proceed inductively on n. Since $r \geq 2$ there is an exact sequence

$$0 \to \Lambda^{r-1} \mathcal{O}_X(-1)^{\oplus n} \to \Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} \to \Lambda^r \mathcal{O}_X(-1)^{\oplus n} \to 0.$$

I obtained the map

$$\Lambda^r \mathcal{O}_X(-1)^{\otimes n+1} \to \Lambda^r \mathcal{O}_X(-1)^{\otimes n}$$

by carefully applying (II, Ex 5.16d) to the map $\mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X(-1)^{\oplus n}$. But this map just turns out to be locally defined by $x_n \mapsto 0$ where $x_0, \ldots, x_n$ are local coordinates for $\mathcal{O}_X(-1)$. Then $x_{i_0} \wedge \cdots \wedge x_{i_r}$ maps to 0 if some $i_k = n$ and itself otherwise. The map

$$\Lambda^{r-1} \mathcal{O}_X(-1)^{\oplus n} \to \Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}$$

identifies $\Lambda^{r-1} \mathcal{O}_X(-1)^{\oplus n}$ with the kernel of the next map. The kernel of the next map is locally generated by all "monomials" which contain an x_n. Since $r \geq 2$ we can identify $\Lambda^{r-1} \mathcal{O}_X(-1)^{\oplus n}$ with this kernel by just removing the x_n off of the wedge product. [This is not rigorous enough!]

By induction on n we have that

$$H^i(\Lambda^{r-1} \mathcal{O}_X(-1)^{\oplus n}) = H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n}) = 0$$

for all i (we will do the base case $n = 1$ in just a moment). Thus, by the long exact sequence of cohomology we see that $H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}) = 0$ for all i. For $n = 1$, since $r \geq 2$ it follows that $\Lambda^{r-1} \mathcal{O}_X(-1) = \mathcal{O}_X(-1)$ or 0 and $\Lambda^r \mathcal{O}_X(-1) = 0$ and these both have trivial cohomology as computed above.

Step 3. The final step is to obtain the long exact sequence

$$\cdots H^i(\Omega^r) \to H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}) \to H^i(\Omega^{r-1}) \to \cdots$$

then apply step 2 and the induction hypothesis (we are inducting on r, the base case was established in step 1) to calculate $H^i(\Omega^r)$ for all i.

Suppose $r \geq 1$, then by (II, 8.13) we have an exact sequence

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0.$$

By (II, Ex. 5.16d), $\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}$ has a filtration

$$\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong \Omega^p \otimes \Lambda^{r-p}\mathcal{O}_X = \begin{cases} 0 & \text{if } r-p \geq 2 \\ \Omega^p & \text{if } r-p \text{ is 0 or 1} \end{cases}$$

Thus

$$\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} = F^0 = \cdots = F^{r-1}$$

and the filtration becomes

$$\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} \supset F^r \supset F^{r+1} = 0$$

with $\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}/F^r \cong \Omega^{r-1}$ and $F^r \cong \Omega^r$. This gives an exact sequence

$$0 \to \Omega^r \to \Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} \to \Omega^{r-1} \to 0.$$

The associated long exact sequence of cohomology gives for each i an exact sequence

$$H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}) \to H^i(\Omega^{r-1}) \to H^{i+1}(\Omega^r) \to H^{i+1}(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}).$$

But by step 2 the groups $H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1})$ all vanish. Thus $H^i(\Omega^{r-1}) \cong H^{i+1}(\Omega^r)$. By induction on r this shows that

$$H^i(\Omega^r) = \begin{cases} 0 & \text{if } i \neq r \\ k & \text{if } i = r \end{cases}.$$

This completes the proof.