On the generation of the coefficient field of a newform by a single Hecke eigenvalue

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Abstract

Let f be a non-CM newform of weight $k \geq 2$ without nontrivial inner twists. In this article we study the set of primes p such that the eigenvalue $a_p(f)$ of the Hecke operator T_p acting on f generates the field of coefficients of f. We show that this set has density 1, and prove a natural analogue for newforms having inner twists. We also present some new data on reducibility of Hecke polynomials, which suggest questions for further investigation.

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1 Introduction

The main aim of this paper is to prove the following theorem.

Theorem 1. *Let f be a newform (i.e., a new normalized cuspidal Hecke eigenform) of weight $k \geq 2$, level N and Dirichlet character χ which does not have complex multiplication (CM, see [R80, p. 48]). Let $E_f = \mathbf{Q}(a_n(f) : (n, N) = 1)$ be the field of coefficients of f and $F_f = \mathbf{Q}\left(\frac{a_n(f)^2}{\chi(n)} : (n, N) = 1\right)$.*

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(a) The set

$$\left\{p \text{ prime} : \mathbf{Q}\left(\frac{a_p(f)^2}{\chi(p)}\right) = F_f\right\}$$

has density 1.

(b) If f does not have any nontrivial inner twists, then the set

$$\{p \text{ prime} : \mathbf{Q}(a_p(f)) = E_f\}$$

has density 1.

A twist of f by a Dirichlet character ϵ is said to be *inner* if there exists a (necessarily unique) field automorphism $\sigma_\epsilon : E_f \to E_f$ such that

$$a_p(f \otimes \epsilon) = a_p(f)\epsilon(p) = \sigma_\epsilon(a_p(f))$$

for almost all primes p. If N is square free, $k = 2$ and the Dirichlet character χ of f is the trivial character, then there are no nontrivial inner twists of f. For a discussion of inner twists we refer the reader to [R80, §3] and [R85, §3].

In the presence of nontrivial inner twists, the conclusion of Part (b) of the theorem never holds. To see this, we let ϵ be a nontrivial inner twist with associated field automorphism σ_ϵ. The set of primes p such that $\epsilon(p) = 1$ has a positive density and for any such p we have $\sigma_\epsilon(a_p(f)) = a_p(f)$. Therefore, $a_p(f) \in E_f^{\langle\sigma\rangle} \subsetneq E_f$ for a set of primes p of positive density.

In the literature there are related but weaker results in the context of Maeda's conjecture, i.e., they concern the case of level 1 and assume that $S_k(1)$ consists of a single Galois orbit of newforms (see, e.g., [JO98] and [BM03]). We now show how Part (b) of Theorem 1 extends the principal results of these two papers.

Let f be a newform of level N, weight $k \geq 2$ and trivial Dirichlet character $\chi = 1$ which neither has CM nor nontrivial inner twists. This is true when $N = 1$. Let $\mathbb{T}$ be the $\mathbf{Q}$-algebra generated by all T_n with $n \geq 1$ inside $\mathrm{End}(S_k(N, 1))$ and let $\mathfrak{P}$ be the kernel of the $\mathbf{Q}$-algebra homomorphism $\mathbb{T} \xrightarrow{T_n \mapsto a_n(f)} E_f$. As $\mathbb{T}$ is reduced, the map $\mathbb{T}_{\mathfrak{P}} \xrightarrow{T_n \mapsto a_n(f)} E_f$ is a ring isomorphism with $\mathbb{T}_{\mathfrak{P}}$ the localization of $\mathbb{T}$ at $\mathfrak{P}$. Non canonically $\mathbb{T}_{\mathfrak{P}}$ is also isomorphic as a $\mathbb{T}_{\mathfrak{P}}$-module (equivalently as an E_f-vector space) to its $\mathbf{Q}$-linear dual, which can be identified with the localization at $\mathfrak{P}$ of the $\mathbf{Q}$-vector space $S_k(N, 1; \mathbf{Q})$ of cusp forms in $S_k(N, 1)$ with q-expansion in $\mathbf{Q}[[q]]$. Hence, $\mathbf{Q}(a_p(f)) = E_f$ precisely means that the characteristic polynomial $P_p \in \mathbf{Q}[X]$ of T_p acting on the localization at $\mathfrak{P}$ of $S_k(N, 1; \mathbf{Q})$

is irreducible. Part (b) of Theorem 1 hence shows that the set of primes p such that P_p is irreducible has density 1.

This extends Theorem 1 of [JO98] and Theorem 1.1 of [BM03]. Both theorems restrict to the case $N = 1$ and assume that there is a unique Galois orbit of newforms, i.e., a unique $\mathfrak{P}$, so that no localization is needed. Theorem 1 of [JO98] says that

$$\#\{p < X \text{ prime } : P_p \text{ is irreducible in } \mathbf{Q}[X]\} \gg \frac{X}{\log X}$$

and Theorem 1.1 of [BM03] states that there is $\delta > 0$ such that

$$\#\{p < X \text{ prime } : P_p \text{ is reducible in } \mathbf{Q}[X]\} \ll \frac{X}{(\log X)^{1+\delta}}.$$

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2 Group theoretic input

Lemma 1. *Let q be a prime power and ϵ a generator of the cyclic group $\mathbb{F}_q^\times$.*

(a) The conjugacy classes c in $\mathrm{GL}_2(\mathbb{F}_q)$ have the following four kinds of representatives:

$$S_a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad T_a = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \quad U_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad V_{x,y} = \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}$$

where $a \neq b$, and $y \neq 0$.

(b) The number of elements in each of these conjugacy classes are: $1, q^2 - 1, q^2 + q$, and $q^2 - q$, respectively.

Proof. See Fulton-Harris [FH91], page 68. □

We use the notation $[g]_G$ for the conjugacy class of g in G.

Proposition 1. *Let q be a prime power and r a positive integer. Let further $R \subseteq \widetilde{R} \subseteq \mathbb{F}_{q^r}^\times$ be subgroups. Put $\sqrt{\widetilde{R}} = \{s \in \mathbb{F}_{q^r}^\times \,:\, s^2 \in \widetilde{R}\}$. Set*

$$H = \{g \in \mathrm{GL}_2(\mathbb{F}_q) \,:\, \det(g) \in R\}$$

and let

$$G \subseteq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^r}) \,:\, \det(g) \in \widetilde{R}\}$$

be any subgroup such that H is a normal subgroup of G. Then the following statements hold.

(a) The group $G/(G \cap \mathbb{F}_{q^r}^\times)$ (with $\mathbb{F}_{q^r}^\times$ identified with scalar matrices) is either equal to $\mathrm{PSL}_2(\mathbb{F}_q)$ or to $\mathrm{PGL}_2(\mathbb{F}_q)$. More precisely, if we let $\{s_1, \dots, s_n\}$ be a system of representatives for $\sqrt{\widetilde{R}}/R$, then for all $g \in G$ there is i such that $g \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i^{-1} \end{pmatrix} \in G \cap \mathrm{GL}_2(\mathbb{F}_q)$ and $\left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right) \in G$.

(b) Let $g \in G$ such that $g \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i^{-1} \end{pmatrix} \in G \cap \mathrm{GL}_2(\mathbb{F}_q)$ and $\left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right) \in G$. Then

$$[g]_G = [g \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i^{-1} \end{pmatrix}]_{G \cap \mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right).$$

(c) Let $P(X) = X^2 - aX + b \in \mathbb{F}_{q^r}[X]$ be a polynomial. Then the inequality

$$\sum_C |C| \;\leq\; 2|\widetilde{R}/R|(q^2 + q)$$

holds, where the sum runs over the conjugacy classes C of G with characteristic polynomial equal to $P(X)$.

Proof. (a) The classification of the finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_q)$ yields that the group $G/(G \cap \mathbb{F}_{q^r}^\times)$ is either $\mathrm{PGL}_2(\mathbb{F}_{q^u})$ or $\mathrm{PSL}_2(\mathbb{F}_{q^u})$ for some $u \mid r$. This, however, can only occur with $u = 1$, as $\mathrm{PSL}_2(\mathbb{F}_{q^u})$ is simple. The rest is only a reformulation.

(b) This follows from (a), since scalar matrices are central.

(c) From (b) we get the inclusion

$$\bigsqcup_C C \subseteq \bigsqcup_{i=1}^{n} \bigsqcup_D D \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right),$$

where C runs over the conjugacy classes of G with characteristic polynomial equal to $P(X)$ and D runs over the conjugacy classes of $G \cap \mathrm{GL}_2(\mathbb{F}_q)$ with characteristic polynomial equal to $X^2 - as_i^{-1}X + bs_i^{-2}$ (such a conjugacy class is empty if the polynomial is not in $\mathbb{F}_q[X]$). The group $G \cap \mathrm{GL}_2(\mathbb{F}_q)$ is normal in $\mathrm{GL}_2(\mathbb{F}_q)$, as it contains $\mathrm{SL}_2(\mathbb{F}_q)$. Hence, any conjugacy class of $\mathrm{GL}_2(\mathbb{F}_q)$ either has an empty intersection with $G \cap \mathrm{GL}_2(\mathbb{F}_q)$ or is a disjoint union of conjugacy classes of $G \cap \mathrm{GL}_2(\mathbb{F}_q)$. Consequently, by Lemma 1, the disjoint union $\bigsqcup_D D \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right)$ is equal to one of

(i) $[U_{a,b}]_{\mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right)$,

(ii) $[V_{x,y}]_{\mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right)$ or

(iii) $[S_a]_{\mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right) \sqcup [T_a]_{\mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right)$.

Still by Lemma 1, the first set contains q^2+q, the second set q^2-q and the third one q^2 elements. Hence, the set $\bigsqcup_C C$ contains at most $2|\widetilde{R}/R|(q^2+q)$ elements. □

3 Proof

The proof of Theorem 1 relies on the following important theorem by Ribet, which, roughly speaking, says that the image of the mod ℓ Galois representation attached to a fixed newform is as big as it can be for almost all primes ℓ.

Theorem 2 (Ribet). *Let f be a Hecke eigenform of weight $k \geq 2$, level N and Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$. Suppose that f does not have CM. Let E_f and F_f be as in Theorem 1 and denote by $\mathcal{O}_{E_f}$ and $\mathcal{O}_{F_f}$ the corresponding rings of integers.*

There exists an abelian extension $K/\mathbf{Q}$ such that for almost all prime numbers ℓ the following statement holds:

> *Let $\widetilde{\mathcal{L}}$ be a prime ideal of $\mathcal{O}_{E_f}$ dividing ℓ. Put $\mathcal{L} = \widetilde{\mathcal{L}} \cap \mathcal{O}_{F_f}$ and $\mathcal{O}_{F_f}/\mathcal{L} \cong \mathbb{F}$. Consider the residual Galois representation*
>
> $$\overline{\rho}_{f,\widetilde{\mathcal{L}}} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E_f}/\widetilde{\mathcal{L}})$$
>
> *attached to f. Then the image $\overline{\rho}_{f,\widetilde{\mathcal{L}}}(\mathrm{Gal}(\overline{\mathbf{Q}}/K))$ is equal to*
>
> $$\{g \in \mathrm{GL}_2(\mathbb{F}) \,:\, \det(g) \in \mathbb{F}_\ell^{\times(k-1)}\}.$$

Proof. It suffices to take Ribet [R85, Thm. 3.1] mod $\widetilde{\mathcal{L}}$. Note that F_f is the field E_f^Γ. To see this, one checks immediately that $F_f \subseteq E_f^\Gamma$ with Γ the group of the field automorphisms associated with the inner twists as in [R80, §3]. On the other hand, let σ be a field embedding $E_f \to \mathbf{C}$ which is the identity on F_f, i.e., on $\frac{a_n(f)^2}{\chi(n)}$ for all n with $(n, N) = 1$. Then $\frac{\sigma(a_n(f))^2}{a_n(f)^2} = \frac{\sigma(\chi(n))}{\chi(n)}$ is a root of unity, and, thus, so is $\epsilon(n) = \frac{\sigma(a_n(f))}{a_n(f)}$. This defines a Dirichlet character ϵ by which f has an inner twist. Hence, $\sigma \in \Gamma$ and $F_f = E_f^\Gamma$.

Ribet does not say explicitly that $K/\mathbf{Q}$ is abelian, but this follows since it is a composite of abelian extensions of K, which are each cut out by a character. □

Remark 1. *The field F_f defined in Theorem 1 is invariant under twisting. More precisely, let ϵ be any Dirichlet character and consider the twisted modular form $f \otimes \epsilon$, the Dirichlet character of which is $\chi\epsilon^2$. Then the Fourier coefficients satisfy $a_n(f \otimes \epsilon) = a_n(f)\epsilon(n)$ and, thus, $\frac{a_n(f\otimes\epsilon)^2}{\chi(n)\epsilon(n)^2} = \frac{a_n(f)^2}{\chi(n)}$.*

Remark 2. *If f in Theorem 1 does not have any nontrivial inner twists, then $K = \mathbf{Q}$ and $F_f = E_f$, since $F_f = E_f^\Gamma$ with Γ the group of field automorphisms associated with the inner twists (see the proof of Theorem 2).*

Theorem 3. *Let f be a non-CM newform of weight $k \geq 2$, level N and Dirichlet character χ. Let F_f be as in Theorem 1 and let $L \subset F_f$ be any proper subfield. Then the set*

$$\left\{ p \text{ prime} : \frac{a_p(f)^2}{\chi(p)} \in L \right\}$$

has density zero.

Proof. Let $L \subsetneq F_f$ be a proper subfield and $\mathcal{O}_L$ its integer ring. We define the set

$$S := \{\mathcal{L} \subset \mathcal{O}_{F_f} \text{ prime ideal} : [\mathcal{O}_{F_f}/\mathcal{L} : \mathcal{O}_L/(L \cap \mathcal{L})] \geq 2\}.$$

Notice that this set is infinite. For, if it were finite, then all but finitely many primes would split completely in the extension F_f/L, which is not the case by Chebotarev's density theorem.

Let $\mathcal{L} \in S$ be any prime, ℓ its residue characteristic and $\widetilde{\mathcal{L}}$ a prime of $\mathcal{O}_{E_f}$ lying over $\mathcal{L}$. Put $\mathbb{F}_q = \mathcal{O}_L/(L \cap \mathcal{L})$, $\mathbb{F}_{q^r} = \mathcal{O}_{F_f}/\mathcal{L}$ and $\mathbb{F}_{q^{rs}} = \mathcal{O}_{E_f}/\widetilde{\mathcal{L}}$. We have $r \geq 2$. Let W be the subgroup of $\mathbb{F}_{q^{rs}}^\times$ consisting of the values of χ modulo $\widetilde{\mathcal{L}}$; its size $|W|$ is less than or equal to $|(\mathbf{Z}/N\mathbf{Z})^\times|$. Let $R = \mathbb{F}_\ell^{\times(k-1)}$ be the subgroup of $(k-1)$st powers of elements in the multiplicative group $\mathbb{F}_\ell^\times$ and let $\widetilde{R} = \langle R, W \rangle \subset \mathbb{F}_{q^{rs}}^{\times}$. The size of $\widetilde{R}$ is less than or equal to $|R| \cdot |W|$. Let $H = \{g \in \mathrm{GL}_2(\mathbb{F}_{q^r}) \,:\, \det(g) \in R\}$ and $G = \mathrm{Gal}(\overline{\mathbf{Q}}^{\ker \overline{\rho}_{f,\widetilde{\mathcal{L}}}}/\mathbf{Q})$. By Galois theory, G can be identified with the image of the residual representation $\overline{\rho}_{f,\widetilde{\mathcal{L}}}$, and we shall make this identification from now on. By Theorem 2 we have the inclusion of groups

$$H \subseteq G \subseteq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^{rs}}) : \det(g) \in \widetilde{R}\}$$

with H being normal in G.

If C is a conjugacy class of G, by Chebotarev's density theorem the density of

$$\{p \,\text{prime} \,:\, [\overline{\rho}_{f,\widetilde{\mathcal{L}}}(\mathrm{Frob}_p)]_G = C\}$$

equals $|C|/|G|$. We consider the set

$$M_{\mathcal{L}} := \bigsqcup_{C} \{p \,\text{prime} \,:\, [\overline{\rho}_{f,\widetilde{\mathcal{L}}}(\mathrm{Frob}_p)]_G = C\} \supseteq \left\{ p \,\text{prime} \,:\, \overline{\left(\frac{a_p(f)^2}{\chi(p)}\right)} \in \mathbb{F}_q \right\},$$

where the reduction modulo $\mathcal{L}$ of an element $x \in \mathcal{O}_{F_f}$ is denoted by $\overline{x}$ and C runs over the conjugacy classes of G with characteristic polynomials equal to some $X^2 - aX + b \in \mathbb{F}_{q^{rs}}[X]$ such that

$$a^2 \in \{t \in \mathbb{F}_{q^{rs}} \,:\, \exists u \in \mathbb{F}_q \;\exists w \in W : t = uw\}$$

and automatically $b \in \widetilde{R}$. The set $M_{\mathcal{L}}$ has the density $\delta(M_{\mathcal{L}}) = \sum_C \frac{|C|}{|G|}$ with C as before. There are at most $2q|W|^2 \cdot |R|$ such polynomials. We are now precisely in the situation to apply Prop. 1, Part (c), which yields the inequality

$$\delta(M_{\mathcal{L}}) \leq \frac{4|W|^3 q(q^{2r} + q^r)}{(q^{3r} - q^r)} = O\left(\frac{1}{q^{r-1}}\right) \leq O\left(\frac{1}{q}\right),$$

where for the denominator we used $|G| \geq |H| = |R| \cdot |\mathrm{SL}_2(\mathbb{F}_{q^r})|$.

Since q is unbounded for $\mathcal{L} \in S$, the intersection $M := \bigcap_{\mathcal{L} \in S} M_{\mathcal{L}}$ is a set having a density and this density is 0. The inclusion

$$\left\{ p \,\text{prime} \,:\, \frac{a_p(f)^2}{\chi(p)} \in L \right\} \subseteq M$$

finishes the proof. □

Proof of Theorem 1. To obtain (a), it suffices to apply Theorem 3 to each of the finitely many sub-extension of F_f. (b) follows from (a) by Remark 2 and the fact that χ must take values in $\{\pm 1\}$, as otherwise E_f would be a CM-field and complex conjugation would give a nontrivial inner twist. □

4 Reducibility of Hecke polynomials: questions

Motivated by a conjecture of Maeda, there has been some speculation that for every integer k and prime number p, the characteristic polynomial of T_p acting on $S_k(1)$ is irreducible. See, for example, [FJ02], which verifies this for all $k < 2000$ and $p < 2000$. The most general such speculation might be the following question: *if f is a non-CM newform of level $N \geq 1$ and weight $k \geq 2$ such that some $a_p(f)$ generates the field $E_f = \mathbf{Q}(a_n(f) : n \geq 1)$, do all but finitely many prime-indexed Fourier coefficients $a_p(f)$ have irreducible characteristic polynomial?* The answer in general is no. An example is given by the newform in level 63 and weight 2 that has an inner twist by $\left(\frac{\cdot}{4}\right)$. Also for non-CM newforms of weight 2 without nontrivial inner twists such that $[E_f : \mathbf{Q}] = 2$, we think that the answer is likely no.

Let $f \in S_k(\Gamma_0(N))$ be a newform of weight k and level N. The *degree* of f is the degree of the field E_f, and we say that f is a *reducible newform* if the characteristic polynomial of $a_p(f)$ is reducible for infinitely many primes p.

For each even weight $k \leq 12$ and degree $d = 2, 3, 4$, we used [SAGE] to find newforms f of weight k and degree d. For each of these forms, we computed the *reducible primes* $p < 1000$, i.e., the primes such that the characteristic polynomial of $a_p(f)$ is reducible. The result of this computation is given in Table 1. Table 2 contains the number of reducible primes $p < 10000$ for the first 20 newforms of degree 2 and weight 2. This data inspires the following question.

Question 1. *If $f \in S_2(\Gamma_0(N))$ is a newform of degree 2, is f necessarily reducible? That is, are there infinitely many primes p such that $a_p(f) \in \mathbf{Z}$, or equivalently, such that the characteristic polynomial of $a_p(f)$ is reducible?*

Tables 4–6 contain additional data about the first few newforms of given degree and weight, which may suggest other similar questions. In particular, Table 4 contains data for all primes up to 10^6 for the first degree 2 form f with $L(f, 1) \neq 0$, and for the first degree 2 form g with $L(g, 1) = 0$. We find that there are 386 primes $< 10^6$ with $a_p(f) \in \mathbf{Z}$ (i.e., has reducible characteristic polynomial), and 309 with $a_p(g) \in \mathbf{Z}$.

Question 2. *If $f \in S_2(\Gamma_0(N))$ is a newform of degree 2, can the asymptotic behaviour of the function*

$$N(x) := \#\{p\,\text{prime} :\ p < x, a_p(f) \in \mathbf{Z}\}$$

be described as a function of x?

The authors intend to investigate these questions in a subsequent paper.

Table 1: Counting Reducible Characteristic Polynomials

k	d	N	reducible $p < 1000$
2	2	23	13, 19, 23, 29, 43, 109, 223, 229, 271, 463, 673, 677, 883, 991
2	3	41	17, 41
2	4	47	47
4	2	11	11
4	3	17	17
4	4	23	23
6	2	7	7
6	3	11	11
6	4	17	17
8	2	5	5
8	3	17	17
8	4	11	11
10	2	5	5
10	3	7	7
10	4	13	13
12	2	5	5
12	3	7	7
12	4	21	3, 7

Table 2: First 20 Newforms of Degree 2 and Weight 2

k	d	N	#{reducible $p < 10000$}
2	2	23	47
2	2	29	42
2	2	31	78
2	2	35	48
2	2	39	71
2	2	43	43
2	2	51	64
2	2	55	95
2	2	62	77
2	2	63	622 (inner twist by $\left(\frac{\cdot}{4}\right)$)

k	d	N	#{reducible $p < 10000$}
2	2	65	43
2	2	65	90
2	2	67	51
2	2	67	19
2	2	68	53
2	2	69	47
2	2	73	43
2	2	73	55
2	2	74	52
2	2	74	21

Table 3: Newforms 23a and 67b: values of $\psi(x) = \#\{\text{reducible } p < x \cdot 10^5\}$

k	d	N	r_{an}	1	2	3	4	5	6	7	8	9	10
2	2	23	0	127	180	210	243	277	308	331	345	360	386
2	2	67	1	111	159	195	218	240	257	276	288	301	309

Table 4: First 5 Newforms of Degrees 3, 4 and Weight 2

k	d	N	reducible $p < 10000$
2	3	41	17, 41
2	3	53	13, 53
2	3	61	61, 2087
2	3	71	23, 31, 71, 479, 647, 1013, 3181
2	3	71	13, 71, 509, 3613

k	d	N	reducible $p < 10000$
2	4	47	47
2	4	95	5, 19
2	4	97	97
2	4	109	109, 4513
2	4	111	3, 37

Table 5: First 5 Newforms of Degrees 2, 3 and Weight 4

k	d	N	reducible $p < 1000$
4	2	11	11
4	2	13	13
4	2	21	3, 7
4	2	27	3, 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127, 139, 151, 157, 163, 181, 193, 199, 211, 223, 229,241, 271, 277, 283, 307, 313, 331, 337, 349, 367, 373, 379, 397, 409, 421, 433, 439, 457, 463, 487, 499, 523, 541, 547, 571, 577, 601, 607, 613, 619, 631, 643, 661, 673, 691, 709, 727, 733, 739, 751, 757, 769, 787, 811, 823, 829, 853, 859, 877, 883, 907, 919, 937, 967, 991, 997 (has inner twists)
4	2	29	29

k	d	N	reducible $p < 1000$
4	3	17	17
4	3	19	19
4	3	35	5, 7
4	3	39	3, 13
4	3	41	41

Table 6: Newforms on $\Gamma_0(389)$ of Weight 2

k	d	N	reducible $p < 10000$
2	1	389	none (degree 1 polynomials are all irreducible)
2	2	389	5, 11, 59, 97, 157, 173, 223, 389, 653, 739, 859, 947, 1033, 1283, 1549, 1667, 2207, 2417, 2909, 3121, 4337, 5431, 5647, 5689, 5879, 6151, 6323, 6373, 6607, 6763, 7583, 7589, 8363, 9013, 9371, 9767
2	3	389	7, 13, 389, 503, 1303, 1429, 1877, 5443
2	6	389	19, 389
2	20	389	389

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