THE JACOBIAN, THE ABEL-JACOBI MAP, AND ABEL'S THEOREM

SETH KLEINERMAN

1. Introduction

Throughout, X will denote a compact Riemann surface of genus $g \geq 1$. Recall that a *divisor* on X is a formal sum of points p in X with integer coefficients,

$$D = \sum_{p \in X} n_p p, \ n \in \mathbb{Z}.$$

Also, any meromorphic function $f : X \to \mathbb{C}$ has a divisor naturally associated to it, namely

$$(f) = \sum_{p \in X} (ord_p(f))p.$$

The *degree* of a divisor D is the sum of its integer coefficients: for D as above,

$$\deg(D) = \sum_{p \in X} n_p.$$

A natural question to ask is: which divisors of degree 0 do not arise from meromorphic functions? The answer is given in a theorem of Abel, which we will present here. Since each divisor up to linear equivalence also corresponds to an isomorphism class of line bundles of degree 0, we will also be able to use Abel's theorem to classify degree 0 line bundles on X as points of a complex torus called the Jacobian.

2. The Jacobian

The first step is to introduce the Jacobian of X, which we will define to be the compact quotient of $\mathbb{C}^g$ by a certain lattice.

To start, consider a canonical basis for the homology group $H_1(X, \mathbb{Z})$. It has $2g$ elements, $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$. Each of these correspond to closed curves in the g-handled torus, with a_i and b_i representing the curves around the inner and outer circumferences of the ith handle.

We will denote the line bundle whose sections are holomorphic 1-forms on X by Ω, and the trivial line bundle (whose sections are holomorphic functions on X) by $\mathcal{O}$. Let $\omega_1, \omega_2, \ldots, \omega_g$ be a normalized basis for $H^0(X, \Omega)$. (By Serre duality, this space is isomorphic to $H^1(X, \mathcal{O})^*$.) The choice of basis is dependent on the homology basis chosen above; the normalization signifies that

$$\int_{a_i} \omega_j = \delta_{ij}, \qquad i, j = 1, 2, \ldots, g.$$

Now for each curve γ in the homology group $H_1(X, \mathbb{Z})$, we can associate a vector λ_γ in $\mathbb{C}^g$ by integrating each of the g 1-forms over γ, as follows:

$$\lambda_\gamma = \left(\int_\gamma \omega_1, \int_\gamma \omega_2, \ldots, \int_\gamma \omega_g \right).$$

Because we have explicitly chosen the ω_is to be normalized with respect to the canonical homology basis, we see that

$$\lambda_{a_i} = e_i,$$

the ith orthonormal basis vector, and we can define

$$\lambda_{b_i} = \left(\int_{b_i} \omega_1, \int_{b_i} \omega_2, \ldots, \int_{b_i} \omega_g \right) = B_i.$$

These $2g$ vectors (the λ_{a_i}s and λ_{b_i}s) are in fact $\mathbb{R}$-linearly independent in $\mathbb{C}^g$, that is, no nontrivial linear combination of the vectors with coefficients in $\mathbb{R}$ can be equal to zero.

Proof. If they were not, for some $s_i, t_i \in \mathbb{R}$ not all zero, we would have

$$\sum_{i=1}^{g} s_i e_i + \sum_{i=1}^{g} t_i B_i = 0.$$

Rewriting this in terms of the original basis, we get

$$\sum_{i=1}^{g} \left(s_i \int_{a_i} \omega_j + t_i \int_{b_i} \omega_j \right) = 0 \text{ for each } j, \text{ so}$$

$$\sum_{i=1}^{g} \left(s_i \int_{a_i} \bar{\omega}_j + t_i \int_{b_i} \bar{\omega}_j \right) = 0 \text{ for each } j \text{ as well.}$$

Since the ω_i and $\bar{\omega}_i$ constitute a basis for the De Rham cohomology $H^1_{DR}(X)$, our $2g$ equations of the form $\int_\gamma \omega_i = 0$ and $\int_\gamma \bar{\omega}_i = 0$ imply that all integrals of elements of the cohomology around γ are 0. By Poincaré duality, the homotopy class of the curve γ must be degenerate, that is,

$$\gamma = \sum_{i=1}^{g} (s_i[a_1] + t_i[b_i]) = 0,$$

where bracketing $[a_i]$ and $[b_i]$ is used to represent their homology classes in $H_1(X, \mathbb{R})$. However, this last result cannot be true since the set $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ was chosen at the outset to be the canonical basis for the homology group $H_1(X, \mathbb{Z})$, and cannot satisfy such an equation. $\square$

Therefore these vectors generate a $2g$-real-dimensional lattice Λ in $\mathbb{C}^g$, which we define by

$$\Lambda = \{s_1 e_1 + \cdots + s_g e_g + t_1 B_1 + \cdots + t_g B_g, \text{ with } s_i, t_i \in \mathbb{Z}\}.$$

The *Jacobian* of the Riemann surface X, denoted $J(X)$, is the compact quotient $\mathbb{C}^g/\Lambda$. We can understand it by analogy with the quotient of a real $2g$-dimensional vector space with a $2g$-dimensional lattice; it is a complex torus.

We can also introduce the Jacobian without relying on a choice of basis, although the resulting structure is less intuitive. Define a map as follows:

$$\begin{aligned} \varphi : H_1(X, \mathbb{Z}) &\longrightarrow H^0(X, \Omega)^* \\ \gamma \qquad &\longmapsto \left(\omega \mapsto \int_\gamma \omega \right). \end{aligned}$$

This map takes closed curves in the homology group to the functionals of integration around those curves, which are dual to the vector space of holomorphic 1-forms. The image of φ is a discrete lattice in $H^0(X, \Omega)^*$, and the Jacobian introduced above can also be defined as the quotient by this lattice: $J(X) = H^0(X, \Omega)^* / \operatorname{Im} \varphi$.

3. The Abel-Jacobi Map

Fix a base point $p_0 \in X$. The *Abel-Jacobi map* is a map $\mu : X \to J(x)$. For every point $p \in X$, choose a curve c from p_0 to p; define the map μ as follows:

$$\mu(p) = \left(\int_{p_0}^p \omega_1, \int_{p_0}^p \omega_2, \ldots, \int_{p_0}^p \omega_g \right) \mod \Lambda,$$

where the integrals are all along c. We have to check that $\mu(p)$ is well-defined as a member of $J(X)$, i.e., that the choice of curve c does not affect the value of $\mu(p)$.

Proof. Choose two curves c, c' from p_0 to p. Notice that $c - c' = \gamma$, a closed curve. We can then write

$$\left(\int_c \omega_1, \ldots, \int_c \omega_g \right) - \left(\int_{c'} \omega_1, \ldots, \int_{c'} \omega_g \right) = \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \right) = \lambda_\gamma.$$

Since γ is a cycle in the homology group $H_1(X, \mathbb{Z})$, this difference is an element of the lattice Λ. (To get the element explicitly, write γ in terms of the canonical homology basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$, then remember that λ_γ is written the same way in terms of the corresponding basis for the lattice, $\{e_1, \ldots, e_g, B_1, \ldots, B_g\}$.) So the map is well-defined. □

After having defined the Abel-Jacobi map on the points of X, it extends to divisors on X by linearity:

$$\mu \left(\sum_{p \in X} n_p p \right) = \sum_{p \in X} n_p \mu(p).$$

4. Abel's Theorem

In the previous section, we defined the Abel-Jacobi map μ, which brings divisors on X to points of the complex torus $J(X)$. Abel's theorem classifies divisors by their images in the Jacobian.

We will need two lemmas; the latter will be stated without proof.

Lemma 4.1. *For two points p and q in X, we can produce a 1-form which has a simple pole at both points, is holomorphic everywhere else in X, and carries a residue of 1 at p and -1 at q. Moreover, we can normalize this 1-form by subtracting a holomorphic 1-form from it; the normalized 1-form, now unique, is denoted ω_{pq}, and the normalization means that*

$$\int_{a_i} \omega_{pq} = 0$$

for $i = 1, 2, \dots, g$.

Proof. Consider the divisor $p+q$. It has a line bundle associated with it, denoted $L(p+q)$. Define $\Omega(p+q)$ to be $H^0(\Omega \otimes L(p+q))$. We want to find the degree of this space. By Riemann-Roch,

$$h^0(\Omega(p+q)) - h^1(\Omega(p+q)) = 1 - g + \deg(\Omega(p+q)).$$

First, it is a well-known result that the degree of Ω is $2g-2$ (the proof is an application of Riemann-Roch and Serre duality), and we know that the degree of $(p+q)$ is 2. Since degrees add, we have $\deg(\Omega(p+q)) = 2g$.

Now $h^1(\Omega(p+q)) = h^0(L(-p-q))$ by Serre duality. But $L(-p-q)$ corresponds to the space of meromorphic functions f on X such that $\text{div} f - p - q \geq 0$, that is, holomorphic functions on X which have zeroes at p and q. But all the holomorphic functions on X are the constant functions, so the dimension of this space is 0.

Putting these results together, the equation above now reads

$$h^0(\Omega(p+q)) - 0 = 1 - g + 2g,$$

and we have shown that $h^0(\Omega(p+q)) = g+1$. Also since the degree of (p) is 1, the same argument as above shows that $h^0(\Omega(p)) = g$, which is the same as $h^0(\Omega)$. So in order to account for this increase in degree, we must conclude that there exists some meromorphic form with simple poles at p and q.

Since it is generally known that the sum of the residues of a form on a compact Riemann surface is 0, we can scale this form by a constant to get residues of 1 and -1 at p, q respectively. Since the space of such forms had dimension 1, this scaling produces a unique 1-form, which completes the proof of the lemma. $\square$

Lemma 4.2 (Reciprocity Law). *Let* $\omega_1, \omega_2, \dots, \omega_g$ *be a normalized basis for* $H^0(X, \Omega)$ *as before. Then*

$$\int_{b_k} \omega_{pq} = 2\pi i \int_q^p \omega_k.$$

Note that the right-hand integral does not appear to be well-defined; we therefore have to specify that it be taken along a curve from q *to* p *that lies within* X *depicted as a planar* $4g$*-gon before identifications.*

Theorem 4.3 (Abel). *Let* D *be an divisor of degree* 0 *on* X. *Then* D *is the divisor of a meromorphic function* f *if and only if* $\mu(D) = 0$ *in the Jacobian* $J(X)$.

Proof. The divisor D is of degree 0, so we can write it

$$D = \sum_{k=1}^{r} (p_k - q_k),$$

with no points p_i, q_j in common.

Suppose D is the divisor of a meromorphic function f. Consider the 1-form $\frac{df}{f}$. It has a simple pole at every point at which f has a zero or a pole. To see this, take the derivative using the local coordinate wherever f looks like z^n for some $|n| \geq 1$, and then divide by f: the form is explicitly $\frac{n}{z}$. We also see that the residue at this pole of the 1-form is the degree of the zero or pole of f.

So the 1-forms $\frac{df}{f}$ and $\sum_{k=1}^{r} \omega_{p_k q_k}$ have simple poles in the same places and the same residues at those poles; by Lemma 4.1, they differ by some holomorphic

1-form, which we can write in terms of the ω_i basis with coefficients $t_i \in \mathbb{C}$:

$$\frac{df}{f} - \sum_{k=1}^{r} \omega_{p_k q_k} = \sum_{i=1}^{g} t_i \omega_i.$$

The next thing we need to notice is that for γ not containing any points p_i or q_i, the integral $\int_\gamma \frac{df}{f}$ equals $2\pi i m$ for some $m \in \mathbb{Z}$. This observation is intuitive, as follows: for any sufficiently small segment of the curve γ, with endpoints a, b, we can choose a branch of the natural logarithm function; having made that choice, the form $\frac{df}{f} = d(\log f)$ is exact, so the integral from one endpoint of the segment to the other is $\log b - \log a$. In the next segment from b to c, we will add $\log c - \log b$, using a different choice of logarithmic branch. So as we go around the curve γ, the value of the integral will be the sum of the successive differences accumulated by the two different branch choices at each endpoint of each segment. As such, it will be an integer multiple of $2\pi i$.

In particular, since the above statement is true for all curves γ, note that it suffices that the integrals around the particular curves a_i, b_i, the basis for the homology group, compute to be elements of $2\pi i\mathbb{Z}$. Summing up the proof thus far, we have found that if D is the divisor of a meromorphic function f, we can find $t_1, \ldots, t_g$ in $\mathbb{C}^g$ such that the integrals

$$\int_{a_i} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right) \text{ and } \int_{b_i} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right)$$

are all elements of $2\pi i\mathbb{Z}$.

Now we will prove the converse of this statement. Assume that we can find such elements t_i of $\mathbb{C}^g$. Define c_k, c'_k to be small circles around p_k, q_k respectively. Then the homology class of any curve γ in $X - \cup_k \{p_k, q_k\}$ is an integral linear combination of the classes of a_i, b_i, c_k, c'_k. We compute

$$\int_{c_k} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right) = 2\pi i,$$

$$\int_{c'_k} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right) = -2\pi i$$

by noting that $\omega_{p_k q_k}$ has residue 1 at p_k and -1 at q_k by definition. So the integral

$$\int_{\gamma} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right) \in 2\pi i\mathbb{Z},$$

as needed.

Using this last equation, we are able to create a function f whose divisor is D. Choose a base point p_0; define

$$f(p) = e^{\int_{p_0}^{p} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right)}.$$

This function is well defined because a different choice of path for the integral would result in the addition of an integral over some closed curve γ, which we have shown above is an integer multiple of $2\pi i$, so would have no effect. Then we compute

$$\frac{df}{f} = d \int_{p_0}^{p} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right) = \sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i,$$

by a variation of the fundamental theorem of calculus. Hence the divisor of f is D, as we needed.

Putting the two halves of the statement together, we see that D is the divisor of a meromorphic function f if and only if there exist $t_i \in \mathbb{C}$ such that

$$\int_{a_i} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right) \text{ and } \int_{b_i} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right)$$

are all elements of $2\pi i\mathbb{Z}$.

The normalization condition on the 1-forms $\omega_{p_k q_k}$ and the properties of the basis ω_i allow the simplification

$$\int_{a_j} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right) = t_j,$$

since the first integral is zero and the second is normalized. To simplify the integral around b_j, we use the reciprocity law (Lemma 4.2) on the first integral, and the definition of B_i for the second integral:

$$\int_{b_j} \left(\sum_{k=1}^{r} \omega_{p_k q_k} + \sum_{i=1}^{g} t_i \omega_i \right) = \sum_{k=1}^{r} 2\pi i \int_{q_k}^{p_k} \omega_j + \sum_{i=1}^{g} t_i B_{ij}.$$

The a_j integrals are elements of $2\pi i\mathbb{Z}$ if and only if there exist integers $n_1, \ldots, n_g$ such that

$$t_j = 2\pi i n_j,$$

and therefore (substituting and dividing by $2\pi i$) the b_j integrals are elements of $2\pi i\mathbb{Z}$ if and only if there exist integers $m_1, \ldots, m_g$ such that

$$\sum_{k=1}^{r} \int_{q_k}^{p_k} \omega_j + \sum_{i=1}^{g} n_i B_{ij} = m_j.$$

Since this last equation holds for all j, we combine the equations to get the vector equality

$$\sum_{k=1}^{r} \int_{q_k}^{p_k} \omega = -\sum_{i=1}^{g} n_i B_i + \sum_{i=1}^{g} m_i e_i$$

where $\omega = (\omega_1, \ldots, \omega_g)$, and this is exactly what we wanted to show: the right-hand side of the equation is an element of the lattice Λ by inspection, and the left-hand side is the image of the divisor D under the Abel-Jacobi map. □

5. Jacobi Inversion

Abel's theorem demonstrated a correspondence between principal divisors and points in the kernel of the Abel-Jacobi map. The Jacobi inversion problem asks whether we can find a divisor that is the preimage for an arbitrary point in the Jacobian. The central theorem in this section will demonstrate that it is possible.

We start with a lemma.

Lemma 5.1. *A holomorphic map $f : M \to N$ between compact connected complex manifolds of the same dimension is surjective if the Jacobian matrix of the map has nonzero determinant at some point of M.*

Proof. Because the Jacobian is nonsingular at some point, $\operatorname{Im} f$ contains an open set in N. But it is known that $\operatorname{Im} f$ is a subvariety of N, that is, that it has dimension equal to or lower than the manifold N. Since $\operatorname{Im} f$ contains an open set in N, it cannot have lower dimension; hence the map is surjective. □

With this lemma in mind, we can prove that the Abel-Jacobi map is surjective:

Theorem 5.2 (Jacobi Inversion)**.** *Every point in $J(X)$ is the image under μ of a degree-0 divisor of the form*

$$D = \sum_{i=1}^{g} (p_i - p_0).$$

Proof. Continue to choose a base point $p_0 \in X$ for the Abel-Jacobi map. Consider the space $X^{(d)} = X^d/S_d$, the product of X with itself d times modulo elements of the symmetric group S_d. (Each element of this space consists of exactly d points in X, equivalent up to ordering.) Define $\mu^{(d)}$ on $X^{(d)}$ to be given by

$$\mu^{(d)}\left(\sum_{i=1}^{d} p_i\right) = \mu\left(\sum_{i=1}^{d}(p_i - p_0)\right).$$

Under this reformulation, the Jacobi Inversion Theorem is equivalent to the statement that the map $\mu^{(g)}$ is surjective, where g denotes the genus of X as usual.

Consider $D = \sum p_i$ a point of $X^{(g)}$, with all p_i distinct. We have local coordinates $z_1, \dots, z_g$ on X centered at $p_1, \dots, p_g$ respectively, since X is a Riemann surface; we then also have a local coordinate $(z_1, \dots, z_g)$ of $X^{(g)}$ centered at D.

Now we compute the Jacobian matrix of $\mu^{(d)}$ near the divisor D. (With all these Jacobians floating around, one might get confused—rest assured that we are going to find the standard Jacobian of a function from vector calculus.) If D' is a divisor close to D, we can write it as the sum of local coordinates $\sum_{i=1}^{g} z_i$. We write out the map in terms of the integrals explicitly and take the partial derivatives of the function with respect to the coordinate system:

$$\frac{\partial}{\partial z_i}(\mu^{(g)}(D')) = \frac{\partial}{\partial z_i}\left(\int_{p_0}^{z_i} \omega_j\right) = \omega_j/dz_i,$$

that is, the function which, when multiplied by the form dz_i, gives the form ω_j. So the Jacobian of the function $\mu^{(d)}$ at D is the matrix

$$\mathbf{J}(\mu^{(\mathbf{d})}) = \begin{pmatrix} \omega_1/dz_1 & \cdots & \omega_1/dz_g \\ \vdots & \ddots & \vdots \\ \omega_g/dz_1 & \cdots & \omega_g/dz_g \end{pmatrix}$$

Our next goal is to show that we can find a point D at which the above matrix is upper triangular with nonzero diagonal. If we change the local coordinate z_i, the ith column of the Jacobian matrix is multiplied by some nonzero scalar, but the rank of the matrix does not change. So we can put this matrix in upper triangular form without changing its rank, as follows: choose p_1 to be some point where ω_1 is nonzero. Then subtract some scalar times ω_1 from each of the forms $\omega_2, \dots, \omega_g$, so that these forms are all 0 at p_1. This procedure is a change in the local coordinate z_1. Now repeat this method, choosing a point p_2 where ω_2 is nonzero under this new coordinate and subtracting a multiple of ω_2 from $\omega_3, \dots, \omega_g$ to make them 0 at p_2; continuing, we finally find a set of points $p_1, \dots, p_g \in X$ and a modified local coordinate such that the Jacobian matrix is upper triangular with nonzero diagonal.

So there is a point D at which the Jacobian matrix of $\mu^{(d)}$ is nonsingular, and by Lemma 5.1 the Abel-Jacobi map must be surjective.

□

The Jacobi Inversion theorem in conjunction with Abel's theorem implies that there is an isomorphism between the space of divisors of degree 0 on X modulo equivalence (the space $\mathrm{Pic}^0(X)$) and the complex torus $J(X)$. (The Jacobi Inversion theorem demonstrates that the Abel-Jacobi map is surjective, and Abel's theorem tells us that its kernel is precisely those divisors that are linearly equivalent to 0.) Since there is already a correspondence between line bundles of degree 0 on X and elements of $\mathrm{Pic}^0(X)$, we have shown that the complex torus $J(X)$ is a moduli space of line bundles of degree 0 on X.

6. Acknowledgements

My utmost thanks to Alina Marian, who explained it to me.

References

[1] C.H. Clemens. *A Scrapbook of Complex Curve Theory, second edition*. American Mathematical Society, 2003.
[2] H.M. Farkas and I. Kra. *Riemann Surfaces*. Springer-Verlag, 1980.
[3] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, 1978.
[4] R. Narasimhan. *Compact Riemann surfaces*. Birkhäuser, 1992.