EGA V: §1 and §§2.15, 2.16

(formerly numbered as EGA IV: §16 and §§17.15, 17.16)

Edited translation of of Grothendieck's 'prenotes' for EGA V

by

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§**V.1** (former IV 16.) **Singular and supersingular set of a function and differential criteria**

This section will be used in V 5 (former IV 20) on hyperplane sections, but its natural place seems to me to be here.

Definition 1. Let X be a regular prescheme, and ϕ a section of $\mathcal{O}_X$. A point $x \in X$ is called a *singular zero* (or root) of ϕ if we have $\phi_x \in m_X^2$. It is called a *supersingular* zero if it is a singular zero and if in addition the element of $m_X^2/m_X^3 \cong \mathrm{Sym}(m_X/m_X^2)$ which it defines, interpreted as a quadratic form on the dual t_x of m_X/m_X^2 over $k(x)$, is degenerate. (A singular zero (or root) which is not supersingular is sometimes called an *ordinary singular zero.*)

Remark 2. If $x \in V(\phi)$, then x is a non-singular zero of ϕ if and only if $\phi_x \neq 0$ and x is a non-singular point, i.e. a regular point of $V(\phi)$, i.e. if and only if x is a regular point of $V(\phi)$ and $V(\phi) \neq X$ in a neighborhood of x.

Definition 3. Let X be a smooth prescheme over a field k, ϕ a section of $\mathcal{O}_X$, $x \in V(\phi)$. We say that x is a *geometrically singular* (resp. *geometrically supersingular*) zero of ϕ relative to k, if for every extension k' of k and every point ξ of X with values in k, localized at x, the corresponding point x' of X'_k is a singular (resp. supersingular) zero of ϕ'_k.

Remarks 4.

a) From the criterion that will be developed below, it follows that in Definition 3, it suffices to test with a single point with values in some k' – for example one can take $k' = k(x)$ or $\overline{k(x)}$ and the canonical point with values in this k'.

b) It follows from Remark 2 that x is geometrically non-singular for ϕ if and only if $\phi_x \neq 0$ and $V(\phi)$ is *smooth* over k at x.

c) Suppose we have a prescheme X smooth over another prescheme Y, a section ϕ of $\mathcal{O}_X$, and an $x \in V(\phi)$. Then we say that x is a singular (resp. supersingular) zero relative to Y if it is a singular (resp. supersingular) zero relative to $k(s)$ over the fiber X_s (where s is the image of x in Y).

d) Under the conditions of Definition 1, we see at once that the singularity resp. supersingularity of an $x \in V(\phi)$ for ϕ is not modified if we replace ϕ by $\phi' = u\phi$ where u is a unit at x. It follows immediately that Definition 1 and consequently also Definition 3 can be extended in an obvious way to the case where ϕ is a section of an invertible module L (in such a way as to recover the original definition when $L = \mathcal{O}_X$).

Let X be a prescheme which is smooth over another prescheme Y, and let ϕ be a section of O_X, giving a section $d^2_{X/Y}\phi$ of $P^2_{X/Y}$, which reduces to a section of $d^1_{X/Y}\phi$ of $P^1_{X/Y}$, which itself reduces to the section $d^0_\phi = \phi$ of $P_{X/Y} = O_X$.

Proposition 5. *The set of zeros of d^0_ϕ (resp. d^1_ϕ) is equal to the set $V(\phi)$ of zeros of ϕ (resp. to the set $V(\phi)^{\text{sing}}$ of zeros of ϕ singular relative to S).*

The first assertion is trivial. The second one is just the Jacobian criterion, or if one prefers, it follows from the canonical isomorphism $m_X/m_X^2 \cong \Omega^1_{X/k}(x)$ which exists when x is a rational point over k of a prescheme X over k.

Note that $\mathrm{gr}^1(P^1_{X/Y}) \cong \Omega^1_{X/Y}$ so that consequently the restriction $d^1\phi \mid V(\phi)$ can be interpreted as a section of $\Omega^1_{X/Y} \otimes \mathcal{O}_{V(\phi)}$, which is just the restriction of $d_{X/Y}\phi$ to $V(\phi)$. We can therefore consider the *prescheme of zeros of this section*, which we denote $V(\phi)^{\text{sing}}$, and whose underlying set is just the set of zeros of ϕ singular relative to Y, by Proposition 5. (N.B. If ψ is a section of a locally free module E of finite type over a prescheme X, one can define in an obvious way the sub-prescheme of zeros of ψ, for example as defined by the image ideal of the map $\check{E} \to \mathcal{O}_X$ given by the transpose of ψ. When $E = \mathcal{O}_X^n$ and $\psi = (\psi_1, \ldots, \psi_n)$, then this ideal is just $\Sigma\psi_i\mathcal{O}_X$, which defines $V(\psi_1, \ldots, \psi_n)$. Now, taking the restriction $d^2\phi \mid V(\phi)^{\text{sing}}$ and noting that $\mathrm{gr}^2(P^2_{X/Y}) \cong \mathrm{Sym}^2(\Omega^1_{X/Y})$, we find a canonical section $M(\phi)$ of $\mathrm{Sym}^2(\Omega^1_{X/Y}) \otimes \mathcal{O}_V^{\text{sing}}$. Taking points of X with values in fields, one sees immediately that this section is precisely the one which determines the quadratic forms given in Definition 1 (in the case where X_k is deduced from X/S by $\mathrm{Spec}(k) \to S$). One can deduce a description of the set $V(\phi)^{\text{sup sing}}$ in terms of this section as follows: interpreting $M(\phi)$ as defining a homomorphism

$$M(\phi)' \colon G_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_V^{\text{sing}} \to \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_V^{\text{sing}}$$

take the set of points at which this homomorphism is not an isomorphism. This shows in particular that $V(\phi)^{\text{sup sing}}$ is a closed set. We can make the latter more precise by introducing

$$D(\phi) = \det\, M(\phi) \in \Gamma(\Omega^d_{X/Y})^{\otimes 2} \otimes \mathcal{O}_V^{\text{sing}},$$

and supposing that X has relative dimension d over Y at every point. One could use $V(\phi)^{\text{sup sing}}$ to denote the closed subscheme of $V(\phi)^{\text{sing}}$ (therefore of X) defined by the vanishing of this section (now of an invertible module), whose underlying set is the right one. It would be a good thing to summarize the above construction into a **Proposition 6**.

In the general case, we cannot say anything more precise about $V(\phi)^{\text{sing}}$ and $V(\phi)^{\text{sup sing}}$. Let us now examine a special case which is interesting for certain applications. Assume that Y is also smooth over a prescheme S, with constant relative dimension m to fix our ideas. Assume also that $V(\phi)^{\text{sing}}$, which we denote by V' for simplicity, defined by the vanishing of the section d^1 of the locally free module $P^1_{X/Y}$ of rank $d+1$, is smooth over S of relative dimension $(m+1)-(d+1)=m-1$. (N.B. Note of course that the notations $V(\phi)^{\text{sing}}$ and $V(\phi)^{\text{sup sing}}$ are ambiguous in the sense that there does not intervent the prescheme to which they are related; in the actual case it is assumed (sous entendu Fr) that it is Y and we also notice that it follows form the assumptions tht every singular zero of ϕ in non-singular relative to S. In this situation we can write down the following diagram of locally free (*sheaves*) of modules over V^1):

$$
\begin{tikzcd}
 & & 0 & & \\
 & & \Omega^1_{X/Y}\otimes\mathcal{O}_{V^1} \arrow[u] & & \\
0 \arrow[r] & P^{V_1}_{X/Y}\otimes\mathcal{O}_{V^1} \arrow[r] \arrow[ur, "\mu"] & \Omega^1_{X/S}\otimes\mathcal{O}_{V^1} \arrow[r] \arrow[u] & \Omega^1_{V^1/S} \arrow[r] & 0 \\
 & \mathcal{O}_{V^1} \arrow[r] \arrow[u] & \Omega^1_{Y/S}\otimes\mathcal{O}_{V^1} \arrow[u] \arrow[ur, "\nu"'] & & \\
\omega^{-2}_{X/Y}\otimes\mathcal{O}_{V^1} \arrow[ur] & & 0 \arrow[u] & &
\end{tikzcd}
$$

The columns come from the exact sequence of transitivity for the smooth morphisms $X\to Y$ and $Y\to S$ and tensoring with $\mathcal{O}_{V'}$ (this remains exact since all the modules in the sequence are locally free). The horizontal line is a particular case of an exact sequence obtained every time when over X over S we have a section ψ of a locally free module F and if we take the scheme of zeros W we find an exact sequence

$$F^v\otimes O_X\to\Omega^1_{X/S}\otimes O_W\to\Omega^1_{W/S}\to 0$$

and if X/S is smooth the first homomorphism is injective exactly at the point where W is smooth over X with a "good" relative dimension (i.e. everywhere in the present case). This exact sequence is an immediate consequence of the exact sequence

$$J/J^2\to\Omega^1_{X/S}\otimes\mathcal{O}_W\to\Omega^1_{X/S}\to 0$$

which appears in Par. 16 (we could state [mettre en corollaire] the version mentioned here).

The characterization of the set of points where we can set a zero on the left is contained is the Jacobian criterion.

Let us note that we have a canonical isomoprhism $P^1_{X/S} = \Omega^1_{X/S} + O_X$ hence $P^{V1}_{X/S} = G_{X/S} + O_X$ (in the original the G is elongated). * On the other hand, we verify that the composed homomorphism μ of the diagram 1 is zero on the factor $O_{V'}$ and on the factor $G_{X/S} \otimes \mathcal{O}_{V1}$ it reduces to the homomorphism $M(\phi)$ (illegible) deduced from the section $M(\phi)$ of Sym 2 $(\Omega^1_{X/S}) \otimes \mathcal{O}_{V'}$ already mentioned. Thus at point x of X, $M(\phi)$ is non-degenerate, i.e. $M(\phi)'$ is surjective if and only if M is surjective at x and we see that in the diagram 1 this is also equivalent to saying that V is surjective at x (since one and the other mean that the canonical homomoprhism of the sum of the two mentioned submodules of $\Omega^1_{X/S} \otimes \mathcal{O}_{V'}$ into the latter is surjective at x.)

We find therefore:

Proposition 7. *Under the preceding conditions (to be recalled) the underlying set of $V(\phi)^{\text{sup sing}}$ is nothing else but the set of points of $V(\phi)^{\text{sing}}$ where the morhism $V(\phi)^{\text{sing}} \to Y$ (of smooth preschemes over S of relative dimension $m-1$ and m respectively) is* ramified.

In the language of the fathers (en termes de papa Fr) (which we should give as a remark) a point $X \in V(\phi)$ is thus supersingular relative to Y if and onlyif "it consists of at least two coinciding (infinite near) singular points (confondus Fr)...

We may and we have to make precise Proposition 7 from the point of view of an identity of sub-preschemes and not just of subsets. Indeed, $V(\phi)^{\text{sup sing}}$ has been defined as a closed pre-subscheme of X or (Fr) we could equally well define a natual closed subscheme of V in such a way that the underlying subset should be the set of ramification points with respect to Y. Indeed it is enough to express the set of points where a certain homomorphism of locally free modules $Q = \Omega^1_{Y/S} \otimes \mathcal{O}_{V'} \to M(= \Omega^1_{V'/S})$ is not surjective. If q and r are their respective ranks this is also the set of points where $\Lambda^1 Q \to \Lambda\sqrt{M}$ is not surjective this is also the zero set of the evident section of $\mathrm{Hom}(\Lambda^1 Q, \Lambda\sqrt{\ } M) \simeq (\approx)(\Lambda\sqrt{\ } Q) \otimes (\Lambda^r R) \otimes (\Lambda^1 Q)^v$, thus the underlying set of a closed sub-prescheme of zeros of this section, let us call it $\mathrm{Ram}(V'/Y)$. I say that the latter subscheme is identical to $V(\phi)^{\text{sup sing}}$. This is a simple exercise about the diagram above, taking into account that $V(\phi)^{\text{sup sing}}$ is defined by the same procedure as the one made explicit for $Q \to R$ but in terms of the homomoprhism $P(= P^{v1}_{X/Y} \otimes \mathcal{O}_{V'}) \to S(= \Omega^1_{X/Y} \otimes \mathcal{O}_{V'})$ as follows from the description of μ given above. We are therefore reduced to the following general situation:

We have on a ringed space W a locally free module M of rank m and two locally free submodules P and Q of respective ranks p and q such that $p+1 = m+1$, we use the previous construction relative to morphisms $P \to M/W = S$ and $Q \to M/P = R$ to find the sections a) of

$$P \otimes \det\, S \otimes \det\, P^{-1} = P \otimes \det\, M \otimes \det\, P^{-1} \otimes \det\, Q^{-1}$$

* Is the elongated G the tangent sheaf? [Tr]

and b) of

$$= Q \otimes \det M \otimes \det P^{-1} \otimes \det Q^{-1}$$

which we may also consider as homomorphisms of $L = \det P \otimes \det Q \otimes \det M$ into P respectively Q. (Nota bene: we denote for a locally free module F by det F its highest exterior power and we use the fact that for a short exact sequence

$$0 \to F^1 \to F \to F^{11} \to 0$$

of such modules we hae a canonical isomoprhism

$$\det F = \det F^1 \otimes \det F^{11},$$

This being given [Fr], we have the *commutativity of the diagram*