Alexandre Grothendieck's EGA V

Translation and Editing of his 'prenotes'

by

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Summary

This formulation gives pêle mêle a detailed summary of the set of results that should appear in a final formulation. To arrive at the latter we need to reorganize thoroughly the present stage zero. The first step should probably be to make a new plan (in which without a doubt the present sections 11, 12, 14, 15 will come much earlier). I have not even written section 16 which should neither in principle cause any difficulty nor does it influence in any way the previous Nos. since what is involved is a simple matter of translation.

You will notice the presence of a proposition 10.3 which should appear in a previous paragraph.

I would like to tell you in this connection that I have several other results quite diverse but all dealing with birational mappings that I would love to include somewhere.

It seems to me that there is not enough to make a paragraph. Do you have a suggestion where to place them?

I plan to send them to you soon as well as section 16 of the present notes.*

In addition, the present paragraph 20 will probably blow up into two paragraphs, one consisting of results of the type “elementary geometry” on grassmanians.

If need be, could one include there also (lacking a better place) the supplements that I told you about dealing with birational transformations?

*Ask AG if No. 16 has ever been written. [Tr]

Hyperplane Sections and Conic Projections

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Hyperplane Sections and Conic Projections

§1 Preliminaries and Notation

Let S be a prescheme, let E be a locally free module of finite type over S, let E^v be its dual. We denote by $P = P(E)$ the projective fibration defined by E and by P^v the projective fibration defined by E^v. We shall call P^v the *scheme of hyperplanes* of p. This terminology can be justified as follows. Let ξ be a section of P^v over S which is threfore determined by an invertiblel quotient module L of E^v. From it we obtain an invertible quotient module L_P of $(E^v)_P = (E_P)^v$, on the other hand, we have the invertible quotient module $O_p(1)$ of E_p. Passing to duals we may take $L_{P^{-1}}$ resp. $O_P(-1)$ to be invertile submodules (locally direct *factors*) of E_P (resp. of $(E_P)^v$) and the pairing $E_P \otimes E_{P^v} \to O_P$ deifnes therefore a natural pairing

$$(*) \qquad O_P(-1) \otimes L_{P^{-1}} \longrightarrow O_P$$

or also the transpose homomorphism

$$(**) \qquad O_P \longrightarrow O_P(1) \otimes L_P = L_P(1)$$

i.e. a section of $L_P(1)$ canonically defined by ξ. The "divisor" of that section, i.e. the closed subscheme H_ξ of P defined by the image ideal of $(*)$, is called the hyperplane in P defined by the element $\xi \in PV(S)$. We could describe it by noting that locally over S, ξ is given by a section ϕ of E such that $\phi(s) \neq 0$ for all s (ϕ is determined by ξ up to multiplication by an invertible section of O_S); since $E = p^*(O_p(1))$, ($p\colon p \to S$ being the projection), ϕ can be considered as a section of $O_p(1)$, the divisor of which is nothing else but H_ξ.

Of course, if we consider L^{-1} as an invertible submodule of E locally a direct factor in E then the correspondence between ξ (i.e. L or $L^{-1} \subset E$) and ϕ is obtained by taking for ϕ a section of L^{-1} which does not vanish at any point, i.e. by a trivialization of L^{-1} (which exists in every case locally). Let us note that H_ξ is nothing else by $P(E/L^{-1})$ (canonical isomorphism) that is a third way of describing H_ξ (N.B. $P(E/L^{-1})$ is indeed canonically embedded in $P = P(E)$ which has the advantage of proving in addition that H_ξ is a projective fibration over S and is a fortiori smooth over S. (Again it would be necessary to say in par. 17 of EGA IV that a projective fibration is smooth.). Without a doubt it would be better to begin with this one.

Remarks. The formation of H_ξ is compatible with base change, in other words one finds a homomorphism of functors $(\mathrm{Sch}/S^0) \to (\mathrm{Ens})$, $P^v \to \mathrm{Div}(P/S)$ where the second term denotes the functor of "relative divisors" of P/S whose values at S' (an arbitrary S prescheme) is the set of closed subschemes of $P_{S'}$ which are complete intersections transversal and of codimension 1 relative to S' (compare Par. 19) [of EGA IV Tr.].[1]

It is easy to show that this homomorphism of functors is a monomorphism, in other words that ξ is determined of H_ξ. (This last fact justifies the terminology "scheme of hyperplanes" used above.) We shall see that the functor $\mathrm{Div}(P/S)$ is representable by the prescheme (direct) sum of $P(Symm^k(E^v))$ so that P^v can be identified to an open and closed subscheme of $\mathrm{Div}(P/S)\ldots$[2] (N.B. to tell the truth, the determination of the relative divisors of P/S could be done with the means available right now, using results on Ch. II and could be added as an example to Par. 19 of EGA IV [Tr.].)

Let us now make the base change $S' = P^v \to S$ and let us consider the diagonal section (or "generic section") of $P^v_{S'} = P(E^v_{S'})$ over S': we find a closed subscheme H_S of $P_{S'} = P \times_S P^v$ which is called sometimes the *incidence scheme* between P and P^v defined by the image ideal of the canonical homomorphism

$$O_P(-1) \otimes_S O_{P^v}(-1) \longrightarrow O_{Px_SP^v}$$

from which one sees that it is a projective fibration over P^v, and by symmetry it is also a projective fibration over P. Let us note that one recovers the "special" hyperplanes H_ξ (for ξ a section of P^v over S) by starting out from the "universal hyperplane" H and by taking its inverse image for the base change $S \xrightarrow{\xi} P^v$.

The same remark holds for every point of P^v with values in an arbitrary S-prescheme S' which (considered as a section of $P_{S'}$ over S') allows us to define an $H_\xi \subset P_{S'}$; the latter is nothing else but the inverse image of H by the base change $S' \xrightarrow{\xi} P^v$.

In what follows we assume a prescheme X of finite type over P [Tr][3] and an S morphism $f\colon X \to P$. One of the main objectives of this paragraph is to study for every hyperplane H_ξ of P its inverse image $Y_\xi = f^{-1}(H_\xi) = XX_PH_\xi$ and especially to relate the properties of X and Y_ξ. As usual one also has to consider the $P(S')$, S' an arbitrary S scheme, in this case H_ξ is a hyperplane in $P_{S'}$ and we put again

$$Y_\xi = f^{-1}_{S'}(H_\xi) = X_{S'} \times_{p_{S'}} H_\xi = XX_PH_\xi$$

where the subscript S' denotes as usual the effect of the base change $S' \to S$ and where in the last expression we consider H_ξ as a P scheme via the combined morphism $H_\xi \to P_{S'} \to P$. It is therefore again convenient to consider the case where ξ is "universal" i.e.

[1] Uses notation of new edition of EGA I [Tr.]
[2] Compare with Mumford's: 'Lectures on curves on an algebraic surface.' [Tr]
[3] or over S, I am not sure [Tr]

where $S' = P^v$ and ξ is the diagonal section so that $H_\xi = H$, in this case one observes (up to better notations to be suggested by Dieudonné) that $Y = Y_\xi$. In the general case of a $\xi\colon S' \to P^v$, one has therefore also $Y_\xi = Y x_{v_P} S'$. Finally if F is a *sheaf of modules*[4] over X we denote by G_ξ its inverse image over Y_ξ by G its inverse image over H so that one also has $G_\xi = G \otimes_{O_P^v} O_{S'}$.

Let us summarize in a small diagram the essentials of the constructions and notations considered.

$$\begin{array}{ccccccc}
F & & & & G & & G_\eta \\
X & \longleftarrow & X x_S P^v & \longleftarrow & Y & \longleftarrow & Y_\eta \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P & \longleftarrow & P x_S P^v & \longleftarrow & H & \longleftarrow & H_\eta \\
\downarrow & & \downarrow & \swarrow & & & \downarrow \\
S & \longleftarrow & P^v & \longleftarrow & & & S'
\end{array}$$

(The squares and diamonds appearing in this diagram are Cartesian). In the next section we will study systematically the following case: S' is the spectrum of a field K and its image in P^v is generic in the corresponding fiber P^v_s. After making the base change $\operatorname{Spec} k(s) \to S$ we are reduced to the case where S is the spectrum of a field k, which is what as assume in the next section. Also the majority of properties studied for X and Y_ξ are of "geometric nature" and therefore invariant under base change, which allows us also (without loss of generality) to limit ourselves to the case where K is algebraically closed or to the case where $K = \overline{k(\eta)}$, η being the generic point of P^v and .$\xi\colon \operatorname{Spec}(K) \to P^v$ is of course the canonical morphism. We also note that for geometric questions concerning X, Y_ξ we can (after making a base change) restrict ourselves to the case of k algebraically closed.

A terminological reminder. If f is an immersion we usually call Y_ξ a hyperplane section of X (relative to the projective immersion f and the hyperplane H_ξ [Tr.]). There is no reason not to extend this terminology to the case of an arbitrary f.

§2 Study of a generic hyperplane section: local properties

Let us recall that now $S = \operatorname{Spec}(k)$, k is a field. If η is a point of P^v and if $\xi\colon \operatorname{Spec} k(\eta) \to P^v$ is the canonical morphism we also write H_η, Y_η, G_η in place of H_ξ, Y_ξ, G_ξ.

In this section (numero) η denotes always the generic point of P^v.

[4] Ask A.G. If module always means coherent or quasi-coherent sheaf of modules.

Proposition 2.1. *Let us assume that X is irreducible. Then Y_η is irreducible or empty and in the first case it dominates X [illegible, ask AG][5] Y [Tr.] is irreducible.*

Indeed, since $H \to P$ is a projective fibration that is also true for $Y \to X$ which implies that Y is irreducible if X is irreducible. So the generic fiber Y_η [Tr.] of Y over P^v is irreducible or empty in the first case its generic point is the generic point of Y which therefore lies over the generic point of X. q.e.d.

Proposition 2.2. *Let Z be a subset of P. Then its inverse image Z_η in H_η is empty if and only if every point of Z is closed. In particular if Z is constructible then $Z_\eta = \phi$ if and only if Z is finite.*

We may suppose that Z is reduced to a single point z and we only have to prove that the image of H_η in P consists exactly of the non-closed points of P. Denoting by X the closure of z and using 2.1 we only have to prove that $Z_\eta = \phi$ if and only if X is finite (X being a closed subscheme of P). Replacing X by $X_{k(\eta)} \hookrightarrow P_{k(\eta)}$ the 'only if' [French 'il faut'] part results from the following fact for which we have to have a reference and which fact deserves to be restated here as a lemma: if Y is *any* hyperplane section of X and if $Y_\eta = \phi$ then X is finite (indeed $X \subset P - H$ is affine and projective...). The 'is sufficient' part one needs is immediate, for example, by noting that Y is a projective fibration of relative dimension $(n-1)$ over X (n being the relative dimension of P and P^v over S), thus X being finite over k, Y is of absolute dimension $n-1$, $\langle n = \dim P \rangle$ thus the morphism $Y \to P^v$ cannot be dominant thus its generic fiber Y_η is empty.

Corollary 2.3. *Let $f\colon X \to P$ be a morphism of finite type and let Z be a constructible subset of X. In order for its inverse image in Y_η to e empty it is necessary and sufficient tha the image $f(Z)$ should be finite. In particular, in order for Y_η to e empty it is necessary and sufficient that $f(X)$ should be finite.*

Corollary 2.4. *Let Z, Z' be two closed subsets of X with Z irreducible, and let Z_η and Z'_η be their inverse image in Y_η. In order to have $Z_\eta \subset Z'_\eta$ it is necessary and sufficient that $f(Z)$ should be finite or that we have $Z \subset Z'$.*

In order that $Z_\eta = Z'_\eta$ it is necessary and sufficient that $f(Z)$ and $f(Z')$ should be finite or that $Z = Z'$.

This is an immediate consequence of 2.3 because we see that $f(Z - Z \cap Z')$ can only be finite if $Z \subset Z'$ or if $f(Z)$ is finite (because if we do not have $z \subset Z'$ then $z - Z \cap Z'$ is dense in Z thus $f(Z - Z \cap Z')$ is dense in $f(Z)$, and if the former is finite and thus closed, being constructible, so is also the latter.

[5] Ask Grothendieck: What is the meaning and role of underlined capital letters, in Section One for example

Corollary 2.5. *To every irreducible component X_i of X such that* $\dim \overline{f(X_i)} > 0$ *we assign its inverse image $Y_{i\eta}$ in Y_η. Then $Y_{i\eta}$ is an irreducible component of Y_η and we obtain in this manner a one-to-one correspondence between the set of irreducible components X_i of X such that* $\dim \overline{f(X_i)} > 0$ *and the set of irreducible components of Y_η.*

Indeed, it follows from 2.3 that Y_η is union of $Y_{i\eta}$ defined above and that the latter are closed and non-empty subsets of Y; they are also irreducible because of 2.1. Finally, they are mutually without an inclusion relation because of 2.4, hence the conclusion.

Let us notice that if $\dim X_i = d_i$ we have $\dim Y_i = d_i - 1$. More generally:

Proposition 2.6. *Let us suppose that for every irreducible component X_i of X we have* $\dim \overline{f(X_i)} > 0$, *i.e. $Y_{i\eta} \neq \emptyset$ or that*[6] *f is an immersion and* $\dim \overline{f(X)} > 0$ *[slightly illegible, (ask AG)]. Then we have* $\dim Y_\eta = \dim X - 1$.

We are reduced to the case where X is irreducible due to 2.5. By the very construction Y_η is defined starting from $X_{k(\eta)}$ as the divisor of a section of an invertible module over $X_{k(\eta)}$ (being the inverse image of $O_P(1)$). On the other hand $X_{k(\eta)}$ is irreducible (because X is such and $k(\eta)$ is a pure transcendental extension of k which one should have indicated at the beginning of the $N^0 \ldots$) and $Y_\eta \neq X_{k(\eta)}$ since the image of Y_η in X (contrary to that of $X_{k(\eta)}$, which is faithfully flat over X) is not equal to X, indeed it does not contain the closed points of X because of 2.3. It follows that $\dim Y_\eta = \dim X_{k(\eta)} - 1 = \dim X - 1$ *(reference needed for the last equality.)* *q.e.d.*

Proposition 2.7. *Let F be a quasi coherent module over X, hence G_η over Y_η. Let Z_i be the associated prime cycles of F such that* $\dim \overline{f(Z_i)} > 0$. *Let $Z_{i\eta}$ be the inverse image of Z_i in Y_η then the $Z_{i\eta}$ are exactly all the prime cycles associated to G_η. Also, their relations of inclusion are the same as among the Z_i.*

The last assertion is contained in 2.4. On the other hand, since $Y \to X$ is a projective fibration, thus flat with fibers (S_1) and irreducible, it follows from par. 3 of EGA IV that the associated prime cycles to the inverse image G of F over Y are the inverse images of the associated prime cycles of F. Hence they are induced on the generic fiber Y_η of Y over P^v, the fact that the associated prime cycles to G_η are the non-empty *inverse images of the Z_i which proves 2.7 by means of 2.3.*

To tell the truth, the passage through Y is unnecessary (not used) (useless), and we can use directly the fact that $Y_\eta \to X$ is flat with fibers (S_1) (and also geometrically regular, i.e. the morphism is regular) and with irreducible fibers (and even geometrically irreducible: they are localizations of projective schemes) remark for the proof of 2.1.

[6] in French, ou que. I think the proper translation is, where [Tr.].

Proposition 2.8. *Let F be coherent over X and let $y \in Y_\eta$, x is its image in X. Let $P(M)$ be one of the following properties for a module of finite type N over a local noetherian ring A:*

(i) coprof $M \leq n$ (ref)
(ii) M satisfies (S_k) (ref)
(iii) M is Cohen-Macaulay
(iv) M is reduced (ref)
(v) M is integral (ref

Then for $G_{\eta,y}$ to satisfy the property P it is necessary and sufficient that F_x should satisfy it.

This follows immediately from results of paragraph 6[7] taking into account that $Y_\eta \to Y$ is a regular morphism so that $O_{X,x} \to O_{Y\eta,y}$ should be regular. Taking into account 2.3, we obtain thus:

Corollary 2.9. *With the notations for 2.8, let Z be the set of $x \in X$ such that $P(F_x)$ is not satisfied. Then in order for G_η to satisfy the condition P at all the points it is necessary and sufficient that $f(Z)$ should be a finite subset of P, or that* dim $\overline{f(Z)} = 0$.

Indeed, 2.8 tells us that $h^{-1}(Z)$ is a P-singular subset of G_η and it is empty if and only if $f(Z)$ is finite by 2.3 (N.B. h denotes the morphism $Y_\eta \to X$; I have just realized that the letter P in 2.8 has been used double).

Corollary 2.10. *Condition for Y_η to be reduced respectively locally integral.*

Corollary 2.11. *Let $y \in Y_\eta$, in order that Y_η should be regular, respectively should satisfy the property R_k (reference) at y (respectively should be normal at y) it is necessary and sufficient that X should satisfy the same property at x. Let Z be the set of those points of X where X is not regular, resp. E_k (resp. normal); for Y_η to be regular resp. R_k (resp. normal) it is necessary and sufficient that $f(Z)$ should be finite, i.e.* dim $\overline{f(Z)} = 0$.

The same proof as 2.8 and 2.9. One must give the different references assuring that Z is closed (because we must know that it is constructible to apply 2.3).

Let us note that in 2.10 and 2.11 we do not talk at all about the corresponding geometric properties; the results described are of 'absolute' nature. We now examine the properties of geometric nature. (One could, if one wanted to, take the opportunity to change the n^0.)

Geometric Properties [Tr.]

[7] Tr: clear up this reference. Is it EGAIV ?

Theorem 2.12. *Suppose that $f: X \to P$ is unramified. Let $y \in Y_\eta$, let x be its image in X. In order for X to be smooth over k at x it is necessary and sufficient that Y_η should be smooth over $k(\eta)$ at y.*

We may assume that k is algebraically closed. If Y_η is smooth over $k(\eta)$ at y it is regular, thus since Y_η is flat over X, X is regular at x (reference), therefore it is smooth over k at x since k is algebraically closed and thus perfect (reference).

For the converse we can (after replacing X by an open neighborhood of x) assume that X is smooth, and (due to the jacobian criterion of smoothness) to be defined in an open subset U of P by p equations as $X = V(f_1, \dots, f_p)$, where the differentials $df_1, \dots, df_p$ are everywhere linearly independent. By introducing the affine coordinates$S_1, \dots, S_n$ in P^v and the affine coordinates $T_1, T_2, \dots, T_n$ in a neighborhood of x (by choosing a hyperplane H^∞ [at infinity] not containing x) Y_η[Tr.] $\hookrightarrow U_{k(\eta)}$ is then equal to $V(f_1, \dots, f_p, \sum S_i T_i - 1)$ and it suffices to verify that the differentials (relative to $k(\eta)$) of $f_1, \dots, f_p$, $(\sum S_i T_i) - 1$ are linearly independent. However, these differentials are nothing else but the sections of $(\Omega^1_{U/k}) \otimes_k k(\eta)$ [Tr.] [illegible] as follows: $df_1, \dots, df_p$, $\sum S_i dT_i$. Since the df_i are linearly independent at every point of U and since the dT_i form a basis of $\Omega^1_{U/S}$ at every point of U and a fortiori a system of generators, we conclude immediately the linear independence of the written down quantities at every point of $U_{k(\eta)}$ at least when $p \leq n$, i.e. if

$$E = \Omega^1_{U/k} \Big/ \sum_{1 \leq i \leq n} O_U df_i \neq 0$$

this is a small lemma about the family of generators a_i, $1 \leq i \leq n$ of a non-zero locally fre module E, thus $\sum S_i a_i$ considered as a section of $E \times_k k(\eta)$ does not vanish at any point. On the other hand, the case $p = n$ is trivial because then $Y_\eta = \emptyset$.

Corollary 2.13. *Let Z be the set of points of X where X is not smooth over k. In order that Y_η should be smooth over $k(\eta)$ it is necessary and sufficient that Z should be finite. In particular, if X is smooth, the same is true about Y_η.*

Follows from 2.12 and 2.3. More generally we obtain:

Theorem 2.14. *Let y be a point of Y_η, x its image in X. Let $P(A, K)$ be one of the following properties for an algebra A local and noetherian over a field K:*

(i) A is geometrically regular over K.
(ii) A is geometrically (R_k) over K.
(iii) A is separable over K.
(iv) A is geometrically normal over K.

Then $P(O_{X,x}, k) \Leftrightarrow P(O_{Y_{\eta,y}}, k(y))$.

Indeed, taking into account par. 6[8] (iii) and (iv) follow from (ii) and 2.8 (ii). On the other hand, (i) has been proven in 2.12 and it remains to deduce (ii) from (i). To do this let Z be the set of points where $X$is not smooth over k, its inverse image Z_η in Y_η is therefore (by 2.12) the set of points of Y_η at which Y_η is not smooth over $K(\eta)$. But the codimension of Z in X is equal to that of Z_η in Y_η at y because of flatness (reference par. 6[9]. Therefore one is $> k$ if the other one is such which completes the proof.

Corollary 2.15. *Let Z be the set of points of X at which X is not smooth over k (respectively is not geometrically R_k over k, respectively is not separable over k, respectively is not normal over k). In order for Y_η to be smooth (respectively geometrically R_k, respectively separable, respectively geometrically normal) over $k(\eta)$ it is necessary and sufficient that Z should be finite.*

From writing up point of view statements 2.14 and 2.15 contain 2.12 and 2.13 (which we could thus dispense with, stating separately) on the other hand the corollary is practically more important than the theorem which one could give in a proposition or a preliminary lemma so that the corollary would be more glorified.

We can give a variant in the case of property (iii):

Corollary 2.16. *(Let us still suppose that f is an immersion and also that F is coherent), under the conditions fo 2.7, in order that Z_i [Tr] should not be immersed it is necessary and sufficient that $Z_{i\eta}$ [Tr] shuld be such. If that is so then the radical multiplicity of F at Z_i at k is equal to that of G_η at Z_i relative to $k(\eta)$.*

The first assertion is contained in the last assertion of 2.7 For the second, since $Y_\eta \to X$ is flat, therefore the functor $F \to G_\eta$ is exact, and we are reduced by restriction to a neighborhood of the general point of Z_i and by a devissage (unscrewing) to the case where $F = O_{Z_i}$ and we may assume $Z_i = X$. Also, we could start by assuming that X is separate over k is reduced to the case of k algebraically closed.[10] Then the asertion is contained in 2.15 (iii). Then we conclude, as usual:

Corollary 2.17. *Let Z be the set of points of X where F is not separable over k (reference). Then G_η is separable over $k(\eta)$ if and only if Z is finite. In particular, if F is separable over k, then G_η is separable over $k(\eta)$.*

[8] Marginal remark X [Tr.] unramified or k of characteristic zero.
[9] of EGA IV, see 6.5 [Tr.]
[10] incomprehensible

Remark 2.18. The key result 2.12 (and its consequences 2.13 and 2.17) become false if we abandon the assumption that f is an immersion, as we see for example by taking for X a purely inseparable covering of P. However, if k is of characteristic zero, 2.12 and 2.17 are valid without assuming that f is an immersion.

Indeed, it suffices to verify this for 2.12 and this follows from 2.11 and from the fact that for an algebraic prescheme in characteristic zero, smooth = regular.

§3 Generic hyperplane section: geometric irreducibility and connectedness

Theorem 3.1 (Bertini-Zariski). *Let us assume* $\dim f(X) \geq 2$ *and that* X *is geometrically irreducible. Then the generic hyperplane section* Y_η *has the same property.*

Let K/k be the function field of X and let $n = \dim P$; introducing the affine coordinates $T_1, \ldots, T_n$ in P (by choosing a hyperplane at infinity H^∞ such that $f(X)$ is not contained in it) and$S_1, \ldots, S_n$ the affine coordinates in P^v, we see that the function field L of Y_η can be identified with the field of fractions of the integral domain $K[S_1, \ldots, S_n]/(\Sigma\, t_i S_i - 1)$ where the $t_i \in K$ are the images of T_i under $f\colon X \to P$. Since $\dim \overline{f(x)} > 0$, the t_i are not all algebraic over k,'a fortiori they are not all zero; let us assume, for example, that $t_n \neq 0$. We realize then immediately that we have $L = K(S_1, \ldots, S_{n-1})$ (pure transcendental extension), $S_n \in L$ being given by the equation $\Sigma\, t_i S_i - 1 = 0$ as a function of theS_i $(1 \leq i \leq n-1)$ and the t_i $(1 \leq i \leq n)$. On the other hand, $k' = k(\eta)$ can be identified with $k(S_1, \ldots, S_n)$ and the canonical inclusion $k' \to L$ can be obtained by sending S_i to S_i [PB: check this!][11] i.e. k' as a subextension of L is the subextension generated by the S_i $(1 \leq i \leq n)$ or what is evidently the same by the S_i $(1 \leq i \leq n-1)$ and by $S_n = a_0 + a_1 S_1 + \cdots + a_{n-1} S_{n-1}$, where $a_0 = t_n^{-1}$, $a_i = -t_i t_n^{-1}$ for $1 \leq i \leq n-1$.

Let us note that the field generated by the a_i and by the t_i is obviously the same, their common transcendence degree is nothing else but the dimension of $f(X)$.

(N.B. It would be appropriate to include this birational description at least as a corollary to 2.1). The proof of 3.1 is thus reduced to that of

Lemma 3.1.1 (Zariski). *(See translator's note at the end of section [Tr]) Let* k *be a field,* K *an extension of finite type over* k*,* m *an integer* ≥ 0*,* a_i $(0 \leq i \leq m)$ *the elements of* K *such that the transcendence degree of* $k(a_0, \ldots, a_m)$ *over* k *is* ≥ 2*. Let* $L = K(S_1, \ldots, S_m)$

[11] Probably S_i to $[S_i]$, equivalence class of S_i in L [Tr.].

and k' be the subfield $k' = k(S_1, \ldots, S_m,\ a_0 + \sum_1^m a_i S_i)$ of$L$ (the S_i being indeterminates). If K is a primary extension of k then L is a primary extension of k'.[12]

This lemma, or lemmata that resemble it like a brother, wander almost everywhere in the literature. That is why I leave it up to you: the choice of the place from where you will copy a proof, i.e. I do not feel inspired to find a proof with my own means.

Corollary 3.2. *Assume that f is unramified or that the characteristic of k is zero, and that the* dim $\overline{f(X)} \geq 2$. *Then if X is geometrically integral the same is true about Y_η. Indeed, geometrically integral = geometrically irreducible + separable.*

Corollary 3.3. *Let us assume that k is algebraically closed and that for every irreducible component X_i of X we have* dim $f(X_i) \geq 2$, *and suppose also that X is σ-connected, where σ is the set of closed subsets Z of X such that* dim $\overline{f(Z)} = 0$ *(i.e. for every such Z, $X - Z$ is connected). Under such conditions Y_η is geometrically connected over $K(\eta)$.*

Indeed, by a lemma that ought to appear in par. 6[13] with Hartshorne's theorem, the hypothesis signifies that we can join any two irreducible components X' and X'' of X by a chain of irreducible components X_0 and $X', \ldots, X_n = X''$ such that two consecutive ones have an intersection $\notin \sigma$ so that the inverse images X'_η andX''_η are joined by a chain of $X_{i\eta}$ which are geometrically connected over$k(\eta)$ by 3.1 and the intersection of two consecutive ones is $\neq \emptyset$ by 2.3.

It follows (since $Y_\eta = X_\eta$ is the union of the $X_{i\eta}$, X_i running through the set of irreducible components of X) that Y_η is geometrically connected over $k(\eta)$, q.e.d.

Translators's note to 3.1.1 This should be compared with Zariski's collected papers (MIT Press) vol. 1, page 174, vol. 2, page 304. Also Zariski-Samuel vol. 1, page 196, vol. 2, page 230 of the GTM Springer edition. Also Jouanolou: Theoreme de Bertini et application Th. 3.6 and Section 6.

§4 Study of a Variable Hyperplane Section: "Sufficiently General" Sections

We return to the general situation of Section 1, S being an arbitrary prescheme. Also, we suppose that$X$ is of finite presentation over S.

In general, let us note that if $P(Z, k)$ is a "constructible" property of an algebraic *prescheme* Z over a field k then the set of $\xi \in P^v$ such that we have $P(Y_\xi, k(\xi))$ is *constructible* as we see by noting that Y_ξ is the fiber over ξ of $Y \to P^v$ which is a morphism

[12]primary extension probably means that the smaller field is algebraically closed in the larger one (or quasi algebraically closed) [Tr]. Jouanolou Th. 3.6 [Tr]

[13]Ask A.G.

of finite presentation and applying par. 9.[14] We have an analogous remark for a property $P(Z, M, k)$ where Z and k are as above and M is a coherent module over Z; if G is in addition of finite presentation over X then the set of $\xi \in P^v$ such that we have $P(Y_\eta, G_\eta, k(\eta))$ is constructible. On the other hand, in the previous two [Tr] sections we have developed in various cases a criterion for the preceding set E to contain the generic point of P^v, S being the spectrum of field k; *being constructible*, it follows that E contains a non-empty open set: this is the passage of a conclusion from generic hyperplane section to the analogous conclusion for "sufficiently general" hyperplane sections.

Let us note in addition that in the case $S = \mathrm{Spec}(k)$ we also have a converse: in order that the generic hyperplane section should have the property P it is necessary and sufficient that the Y_ξ for ξ in a suitable neighborhood of η should satisfy it and it suffices to require for ξ *closed* in P^v (which for ξ k algebraically closed *leads* or *reduces* to considering k-rational points, i.e. hyperplane sections of X itself (without a prior extension to the base field.)(extension prealable Fr)

This follows, indeed, from the constructibility of E and from the fact that P^v is a Jacobson scheme.

Let us give as an example some special cases where the previous considerations apply:

Proposition 4.2. *Let G be a module of finite presentation over X.*

Let P be one of the following properties for a module M over an algebraic scheme Z over a field K;

(i) coprof $(M) \leq n$.
(ii) M satisfies (S_k).
(iii) M is Cohen Macauley.
(iv) M is without embedded prime cycle components.
(v) M is separable over K.

With these notations if E denotes the set of $\xi \in P$ such thatG_ξ satisfies property P then we have: (a) E is a constructible subset of P^v. (b) Let us suppose that $S = Spec(k)$, k a field, and that F satisfies property P. Let us also suppose that in athe case (v) that k is of characteristic 0 or that $f \colon X \to P$ is unramified, then E contains an open and dense suset of P^v.

Proof. (a) follows from the fact that P is a constructible property which we have seen in Par. 9 of EGA IV. As to (b), it follows from the corresponding results of the previous two sections.

[14] of EGA IV. [Tr]

Regrets To (b): suppose more generally that if Z is the set of points of X where F does not satisfy P, we have $f(Z)$ is finite, i.e. dim $\overline{f(Z)} \leq 0$.

Proposition 4.3. *Let P be one of the following properties (for an algebraic prescheme over a field K):*

(i) Z is smooth over K.
(ii) Z satisfies the geometric property (R_k) over K.
(iii) Z is separable over K.
(iv) Z is geometrically normal over K.
(v) Z is geometrically integral over K.
(vi) Z is geometrically irreducible over K.

Let E be the set of $\xi \in P^v$ such that Y_ξ satisfies P. Then: (a) E is a constructible subset of P^v. (b) Let us suppose $S = \operatorname{Spec} k$, k a field and let us suppose in the cases (i) to (v) that k is of characteristic zero and that $f\colon X \to P$ is unramified. Finally, suppose in the cases (v) and (vi) that dim $\overline{f(X)} \geq 2$. *Assume that X satisfies P then E contains a dense open subset of P^v.*

Proof. Proof is identical to that of 4.2. Let us remark that in the cases (i) to (v) it suffices to assume *only* (in (b)) that $f(Z)$ is finite where Z is the set of points of X where P fails.

Corollary 4.4. *Let us consider the property (C_m) "$\bar{Z}$ cannot be disconnected by a closed subset of dimension $\leq m$ (where $\bar{Z}$ is $Zx_K\bar{K}$, $\bar{K}$ the algebraic closure of K)."*

Let E be the set of $\xi \in P^v$ such that Y_ξ over $K(\xi)$ satisfies C_m. Then: (a) E is constructible. (b) Let us suppose that $S = \operatorname{Spec} k$, k a field, and that for every irreducible component X_i of X we have dim $\overline{f(X)} \geq 2$. *Then if X over k satisfies C_{m+1} then E_ξ contains a dense open subset U of P.*

The constructibility is done by AQT[15] This is a fact that one has forgotten in Par. 9 of EGA IV that perhaps it would still be possible to repair (or fix); the part (b) follows in principle from 3.3 except that 3.3 has been announced in an excessively special manner and consequently should be generalized (the proof given being otherwise essentially unchanged). Also there is an error in the statement of 4.4, which is not valid as such if f is quasi-finite; in the general case instead of considering the dimension of the closed subsets of X respectively of Y_ξ it is sufficient to consider the dimension of their images in P respectively in H_ξ. Redactor demerdetur. [Latin] [Tr. Translate]

[15] What is AQT? Ask AG.

§5 Theorems of Seidenberg Type

5.1. In the present section we give conditions under which the set E defined in 4.1 is open. We deal here with properties of P of local nature over X, respectively Y_ξ, such that we can define the set U of $y \in Y$ so that (if ξ denotes the image of y in P^v) Y_ξ satisfies P at the point y (respectively G_ξ satisfies condition P at y). We give first of all the criteria for U to be open by using paragraph 12.[16] As always we have $E = P^v - h(Y - U)$ [Tr] it follows that if U is open and X is proper over S (since h is proper and a fortiori closed) then E is also open.[17]

5.2. As we have seen in No. 1 Y is defined in $Xx_SP^v = X_{P^v}$ as the "divisor" of a section ϕ of $\mathcal{O}_X(1) \otimes_S O_P^v(1)$ which induces for every $\xi \in\ P^v$ a section $\phi\xi$ of $\mathcal{O}_X(1) \otimes_{k(s)} O_P^v(1)(\xi)$ (a sheaf by the way isomorphic non-canonically to $O_X(1) \otimes_{k(s)} k(\xi) = O_{X_{k(s)}}(1)$) such that Y_ξ is nothing else but the "divisor" of this section (N.B. the divisor of a section ϕ of an invertible module L is defined as the closed subscheme defined by the image ideal of $\phi - 1 : L^{-1} \to O$). If F is a sheaf of modules over X then its inverse image over Y, i.e. the inverse image of $F \otimes_{O_S} O_{P^v} = F_{P^v}$ over the subscheme Y of X_{P^v}, is nothing else but the cokernel of the homomorphism $\phi - 1 \otimes id_{F_{P^v}} \colon F_{P^v}(-1,-1) \to F_{P^v}$ where the notation $(-1,-1)$ explains itself as Mike[18] says. Also G_ξ is the cokernel of analogous homorphism $F_{k(\xi)}(-1,-1) \to F_{k(\xi)}$ where ξ is a point of P (and also we have a corresponding interpretation if ξ, instead of being a point of P^v, denotes a point of P^v with values in an S' over S ...)

In general if L is an invertible module somewhere, ϕ a section defining the sub-prescheme $V(\phi)$, then for every module F the inverse image of F *in* $V(\phi)$ can be identified, by the usual abuse of language, to the cokernel of $id_F \otimes$[19]$\colon F \otimes L^{-1} \to F$.

We say that ϕ is F regular if the preceding homomorphism is injective. If we choose an isomorphism of F and $\mathcal{O}_X$, which is possible locally, such that ϕ is identified to a section of O_X, this terminology is compatible with the one that was already introduced elsewhere.

Proposition 5.3. *With the previous notations let U be the set of $x \in X_P$ with image ξ in P^v such that $\phi\xi$ is $F_{K(\xi)}$ regular at x. Then*

(a) If F is of finite presentation and flat realtive to S then U is open and G/U is flat relative to P^v.

(b) For every $s \in P^v$ if η denotes the generic point of P_S^v then U contains $X_{k(\eta)}$.

[16] Locate that reference, most likely EGA IV [Tr], Yes [Tr].
[17] Since Y is proper over X and P^v is separated over S. (Marginal remark [Tr]).
[18] Mike Artin (I presume P.B.)
[19] ??

Proof.

(a) Since F_{P^v} is of finite presentation and flat relative to P^v the conclusion follows from 11.3[20] (and also from $o_{III}\dots$ in the case of locally noetherian S) (of EGA IV [Tr]).

(b) We may suppose $S = \operatorname{Spec} k$. The associated cycles *to* $F_{k(\eta)}$ are (because of Par. 3)[21] the inverse images of associated cycles Z_i to F. If $f(Z_i)$ is finite, then by 2.3 $Z_{ik(\eta)} \cap Y = \phi$ in the contrary case by 2.6; for example, we also have $Z_{ik(\eta)} \cap Y = Z_{ik(\eta)}$ (by reason of dimension; 2.3 which was already involved in 2.6 and is without a doubt a better reason) which proves that ϕ does not vanish over any of the $Z_{ik(\eta)}$ and therefore proves (b).

Corollary 5.4. *Let V be the set of $\xi \in P^v$ such that $\phi\xi$ is $F_{k(\xi)}$ regular. If F is of finite presentation then V is constructible and it contains the generic points of the fibers of P^v over S. On the other hand, if also X is proper over S and F is flat over S, then the set V is open.*

Remark 5.5. Let $\xi \in P^v$ over $s \in S$ and let us suppose tht F_s should be without associated embedded cycles. Then we see immediately that $\xi \in V$ (notation of 5.4) which means also that every irreducible component of $\operatorname{supp} F_{k(\xi)}$ does not lie over H_ξ (and a little less evidently in this criterion we replace $k(\xi)$ by an arbitrary extension of $k(\xi)$.

Let us note that the hypothesis (S_1) aboutF_s which we have just made is satisfied notably if we suppose F_s Cohen-Macauley (a fortiori if F is CM over S); also in this case G_s is CM (since locally it is deduced from $F_{k(s)}$ which is such by dividing by $a\Phi \cdot F_{k(s)}$ where ϕ is $F_{k(s)}$ regular). The same remarks anyway should (and will have to)be made locally above to characterize the points of U (in place of those of V).

Using now 12.1.1 and 12.1.4[22] we obtain:

Theorem 5.6. *Let us assume tht F is of finite presentation flat relative to S. Let P be one of the properties (i) to (viii) of 12.1.1 or (if we assume $F = O_X$) one of the properties (i) to (iv) of 12.1.4 of EGA IV [Tr]. Let U_P be the set of $x \in X_P$ such that if ξ denotes the image of x in P^v the property$P$ shuld be satisfied by G_ξ (resp. Y_ξ) at the point x and such that $\phi\xi$ is $F_{k(\xi)}$ regular at x. Then U_P is open and G/U_P is flat relative to S.*

Indeed, by the very definition we have $U_P \subset U$ (notation of 5.3 (a)) and we apply Par. 12 to $U \to P^v$ and F_{P^v}/U.

Corollary 5.7. *Let us suppose that $F$is of finite presentation flat relative to S, and $\operatorname{supp} F$ proper over S (e.g. X proper over S). Let V_P be the set of $\xi \in P^v$ such that G_ξ*

[20] Find that reference.
[21] illegible, ask AG or figure out – probablyϕ^{-1} [Tr].
[22] Ask AG about reference – probably EGA IV [Tr]. 12.1.4 does not check out [Tr].

(resp. Y) satisfies the property P and that is $F_{K(\xi)}$ regular. Under these conditions V_P is open (and it is also constructible in every case, i.e. without any assumption of flatness or of properness).

It seems to me that from the point of view of presentation we cannot leave 5.6 as is with a simple reference to conditions enumerated in another volume, but it requires an explicit list (i), (ii),... of properties which we have in view. Also remark (in 5.1 perhaps) that the case P = geometrically normal (with $S = \mathrm{Spec}(k)$ for sure)[23] is due to Seidenberg.

§6 Connectedness of an arbitrary hyperplane section

We shall here combine the already known criterion of geometric connectedness of the generic hyperplane section (3.3) with the connectedness theorem of Zariski in order to obtain a connectedness result for an arbitrary hyperplane section:

Proposition 6.1. *We suppose $S = Spec(k)$, k an algebraically closed field [X proper over k suppose][24] that for every irreducible component X_i of X, $\overline{f(X_i)}$ should be of dimension ≥ 2, finally that X cannot be disconnected by a closed subset Z of X such that* $\dim \overline{f(Z)} \leq 0$. *Under such conditions for every $\xi \in P^v$, Y_ξ is geometrically connected.*

Proof. Since none of the $f(X_i)$ is finite we see that every irreducible component Y_i of Y dominates P^v; on the other hand, $Y \to P^v$ is proper (if Y is proper over k, being such over X which is proper over k). On the other hand, by (3.3), the generic fiber Y_η of $Y \to P^v$ is geometrically connected.

Finally, P^v is regular and à fortiori geometrically unibranch. It now suffices to apply 15.6.3[25] (which is variant of the Zariski connectedness theorem) to conclude that *all* the fibers of $Y \to P^v$ are geometrically connected. q.e.d.

Indeed, it is not difficult by a proof of analogous type to generalize 6.1 in the same sense as in 4.4. If you do not want to trouble yourself with this exercise, at least mention it as a remark. To say also that we do not discriminate in 6.1 with regard to hyperplane sections that have an excessive (extra) dimension. (From the planning point of view) it might be clearer to group together all athe connectedness questions (including 3.3 and 4.4) in the same No. (or section).

§7 Application to the construction of hyperplane sections and multisections

[23] or to be sure [Tr].
[24] illegible.
[25] EGA IV [Tr].

of

specified type

7.1. Let us notice that if $S = \mathrm{Spec}(k)$ where k is an infinite field then every non-empty open subset of P^v contains a k-rational point; therefore in the notations of 4.1 if E (defined in terms of a constructible property P) contains the generic point η, it contains a k-rational point and therefore there exists a hyperplane section of X (defined over k) having the property P. On the other hand, S being again arbitrary, it is immediate that for every $s \in S$ and for every point ξ of the fiber P^v_s rational over $k(s)$, there exists a section ξ of P^v on an open neighborhood U of s which passes through ξ_0. If now E is again defined as in 4.1 in terms of a constructible property$P$ and if we have (for example due to No. 5) the fct that E is open, then if $\xi_0 \in E$, then the section ξ is a section of E over U at least if we sufficiently shrink or diminish U. Therefore we may construct a hyperplane section Y_ξ of X over an open neighborhood U of s such that for every $t \in U$ its fiber $Y_{\xi(t)}$ at t satisfies the property P. If we do not have à priori ξ_0 but if $k(s)$ is infinite we may combine the two preceding remarks to obtain a hyperplane section over an open neighborhood of s having the preceding property. Finally, using No. 5, we have a criterion allowing us to assert that (X resp. F being assumed flat over S which allows us to apply loc. cit.) Y_ξ resp.G_ξ is also flat over S. We may therefore, replacing X byY_ξ, iterate the previous construction which allows, for example under certain conditions, to construct closer and closer (by successive approximations ???)[26] a "multisection" of S' of X over an open neighborhood U of the given point s, such that $S' \to U$ should be finite, flat and with fibers satisfying the property P. If $k(s)$ is finite we may be forced or constrained to do an étale and surjective base change $S' \to U$ (U an open neighborhood of s) before being able to apply the preceding constructions; indeed under the conditions from the start of 6.1, if k is finite there does not necessarily exist a rational point over k in the open non-empty set U, but there certainly exists a closed point of U, thus a point with values in a finite extension k' (necessarily separable) of k; when $k = k(s)$, therefore we may, after making a suitable finite étale extension S' over a neighborhood U of s, corresponding to the residual extension k', i.e. such that $S^1_s \xrightarrow{\sim} \mathrm{Spec}(k')$, restrict ourselves to the favorable situation of the unique point $s' \in S'$ over s after a base change $S' \to S$. I must however, note or point out [un remords Fr] a regret to 4.2 and 4.3 which should have been announced in a slightly more general form [at least as a remark]:If we are given an integer m and if we denote by E the set of $\xi \in P^v$ such tht G_ξ, resp. Y_ξ satisfies P exceptovr a closed set of dimension $\leq m$ (i.e. athe set P-singular Z is of dimension $\leq m$). Then

[26]Translator's note: de proche en proche [Fr].

a) E is a constructible subset of P^v and

b) in the case $S = \mathrm{Spec}(k)$, if F, respectively X, satisfies P except over a set of dimension $\leq m+1$, then E contains a non-empty open set.

Proposition 7.2. *Let us assume that X is proper over S and that F is of finite presentation finite and flat over S. Let P be one of the properties (i) to (v) of (4.2) and let m be an integer. Let $S \in S$ and let us suppose that the set Z_s of points of X_s where F_s does not satisfy P is of dimension $\leq m+1$. Then if also $k(s)$ is infinite there exists a neighborhood U of s in E and a section ξ of P^v over Uh aving the following properties: For every$x \in U$ the set of points of $Y_{\xi,s}$ where $G_{\xi(s)}$ does not satisfy P is of dimension $\leq m$ and $\phi_{\xi(s)}$ is $F_{\xi,s}$ regular. Under such conditions the module G_ξ over Y_ξ is flat relative to U. Finally, if $k(s)$ is not supposed infinite, we can again make the previous construction afater an étale extension of the type anticipated in 7.1.*

Propostion 7.3. *Essentially the same. There is no longer an F and assume that X is flat relative to S we refer to properties (i) to (v) of 4.3 in place of those of 4.2 ("but being careful to make the reservation.") $k(s)$ of characteristic zero or $f\colon X \to P$ is an immersion and in the case (v) that for every $s \in S$ [illegible] irreducible component Z of X_s we have* dim $f(Z) \geq 2$. *[Nota Bene: For (v) compare 12.2.1 (x) and (xi) (we can then [illegible] in the other case 4.3 or 12.2.1 (x)] (marginal remark largely illegible in preceding square brackets).*

(Text crossed out)

Proposition 7.4. *Let $g\colon X \to S$ be a flat proper morphism, let $s \in S$, let us put $n =$* dim *X_s and let us suppose that the dimension of the set of points of X_s where X_s is not separable over $k(s)$ is $\leq n$. (for example X_s separable). Then there exists an open neighborhood U of s and an étale finite, surjective morphism S'- - - -$\succ$$U$ such that $X\, x_S\, S'$ admits a section over S'. If $k(s)$ is infinite we may take for S' a closed subscheme of X U.*[27]

Let us assume to start with that $k(s)$ is infinite. We proceed by induction on n, the case $n = 0$ being trivial. Indeed in that case there exists an open neighborhood U of s such that $X|U$ itself is étale, finite and surjective above U as we see by immediate cross references. If $n > 0$, we apply 7.3 for the "separable" property which allows us to replace X by a "hyperplane section" Y having the same properties up to this that n is replaced by $n-1$. If $k(s)$ is not assumed infinite we begin by making an étale base change, it works. (It goes thorugh)

[27]Unclear, ask AG.

Remark 7.5. In particular if X is projective and separable over S it admits locally over S étale multisections. But we note that we can give examples with X proper and smooth (but non-projective) S, where the same conclusion fails. Of course, the projective assumption cannot be weakened in general to an assumption of quasi-projectiveness as we see, for example, by taking X étale non-finite over S ...[28]

§8 Dimension of the set of exceptional hyperplanes

8.1. In the previous sections and notably Sections 2 and 3, we have given statements asserting that the set of $\xi \in P^v$ such that the set of $\xi \in P^v$ such that Y has a certain property P is constructible and that it contains the generic point η or else that the set Z_P of $\xi \in P^v$ "exceptional for P" is constructible and is rare, i.e. that its closure is of codimension ≥ 1. (Nota Bene: we suppose that $S = \text{Spec}(k)$).

In certain cases we can make this statement more precise by giving a better upper bound for this codimension, which is important for certain questions. For example, if we see that this codimension is greater than or equal to two it follows that a "sufficiently general" straight line D of P^v does not intersect Z_P, whence the existence (if k is infinite) of "linear pencils" of hyperplane sections Y_ξ (ξ a geometric point of D) all of which have the property P (see Section No.[29] for examples).

From the *writing up* point of view, since the results of the present No. make more precise some results of the previous sections, the question arises if it is necessary to do this catching up in a separate section (or number) or to give a more precise version gradually as we move along. Redactor decidetur (Latin).[30]

8.2. Let Z be the set of $\xi \in P^v$ such that dim $Y_\xi >$ dim $X - 1$ and let us suppose that for every irreducible component irr X_i of X we have dim $f(X_i) >$[illegible, is it two, ask A.G.] then Z is of codimension two in P^v. This follows from 2.1 and 2.2 (which implies that every irreducible component of Y dominates P^v) and from the dimension theory for the morphism $Y \to P^v$. Starting from this result we may give as a corollary the case where we start a closed subset Z of X and where we consider the dimension fo the inverse images Z_ξ in the Y_ξ ($\xi \in P^v$) and we may even take for Z the set of $\xi \in P^v$ such that there exists an irreducible component of $T_{k(\xi)}$ whose trace on Y_ξ has the greatest dimension (NB we always assume that for every irreducible Z_i of Z we have dim $f(Z_i) > 0$).

[28] Illegible
[29] Section number omitted, ask A.G.
[30] Editor decide.

Finally the most precise statement in this direction and one that results easily from the first announcement (for X irreducible) and from 2.7 is the following modified statement: F being coherent over X, suppose that for every associated prime cycle T for F we have $\dim f(T) > 0$ then the set of $\xi \in P^v$ such that ϕ_ξ is not $F_{k(\xi)}$-regular is (constructible and) of codimension ≥ 2. (The notation for ϕ_ξ is that from No. 5). We can give this as the principal assertion, and announce the previous assertions as corollaries, the proof being or proceeding via one of the corollaries.

Please note that with the preceding notations if $\xi \in P^v - Z$, then for every $y \in Y_\xi$ we have $\mathrm{coprof}_y\, G_{k(\xi)} = \mathrm{coprof}_y\, G_\xi$ and consequently if $\mathrm{coprof}\, F \leq n$ then for $\xi \in P^v - Z$ we have coproof of $T_\xi \leq n$ in particular if F is Cohen-Macauley then for $\xi \in P^v - Z$, G_ξ is Cohen-Macauley. Finally if F is (S_k) we have that G_ξ is (S_{k-1}) for $\xi \in P^v - Z$ (reference O_{IV}).

8.3. We notice that if F is (S_k) for one $\xi \in P^v$ such that ϕ_ξ is f_k (ξ)-regular and G_ξ has a component of codimension ≥ 2[31] even if $F = \mathcal{O}_X, k = 1$, X being geometrically integral of dimension two where ($k = 2$ X being geometrically integral and geometrically normal of dim 3). It is enough to start from a projective integral surface

$$X \subset P^r$$

over k algebraically closed having a point x where$X$ is not Cohen-Macauley, then for every hyperplane pressing through x the corresponding hyperplane section Y_ξ admits x as an associated embedded cycle (respectively, we start from a normal (thus S_2) integral variety $X \subset P^v$ of dimension three having a point $X \in X$ where X is not Cohen-Macauley, then the Y's passing through x are not CM, i.e. they are met (S_2) at x.)

In these examples the set of "exceptional" ξ for the property (S_k) contains the hyperplane of P^v defined by $x \in P$ and it is of codimension one (and not of codimension $\geq$ two) compare 8.5 below for a general precise result in this direction along these lines?

Proposition 8.4. *Let T be a closed subset of X and suppose that codim$(T, X) \geq k$. Then for every $\xi \in P^v$ we have codim$(T_\xi, Y_\xi) \geq k - 1$. Let Z be the set of $\xi \in P^v$ such that codim$(T_\xi, Y_\xi) = k - 1$ (i.e. codim$(T_\xi, Y_\xi) < k$) then Z is a constructible, nowhere dense [rare Fr] subset of P^v, i.e. $\bar{Z}$ is of codimension ≥ 1 in P^v.*

In order for it to be of codimension ≥ 2 it is necessary and sufficient that for every irreducible component T_i of X of codimension equal to k and such that $\dim \overline{f(T_i)} = 0$

[31] Illegible, ask A.G.

there should exist one irreducible component X_j of X such that $\operatorname{codim}(T_i, X_j) = k$ and $\dim f(X_j) - 0$ (or closure crossed out?), i.e. if f is as quasifinite and $k > 0$, que (it???) T does not have isolated points such that $\dim_x X - k$. The first assertion follows immediately from the following lemma 8.4.1 (a) which is a remorseful afterthought to paragraph 5.

Lemma 8.4.1. *Let X be a locally noetherian prescheme, let L be an invertible module over X, ϕ a section of L, $Y = V(\phi)$, T a closed subset of X. Let us assume that codim$(Y, X) \geq k$.*

Then

a) codim$(T \cap Y, Y) \geq k - 1$.

b) In order to have

$$\text{codim}(T \cap Y, Y) = k - 1$$

i.e. codim$(T \cap Y, Y) < k$

it is necessary and sufficient that there should exist an irreducible component T_i of T contained in Y, and such that codim$(T_i, X) = k$ and such that for every irreducible comnponent X_j of X containing T_i and such that

$$\dim O_{X_j, T_i} = \dim O_{X, T_i} \quad (= k)$$

we have

$$X_j \not\subset Y.$$

The verification of this lemma is immediate due to the general facts in O_{IV}, Chapter IV about dimension. With the assumptions of 8.4, by 8.4.1 (b) we see which ones are the exceptional hyperplanes H_ξ. If we exclude the set Z_0 of $\xi \in P^v$ such that there is an irreducible component R of T or of X such that $\dim f(R) > 0$ and such that R_ξ is of "dimension too large" (a set which is of codimension two and in what follows it does not count the exceptional H_ξ are those for which there exsts a T_i with $\operatorname{codim}(T_i, X) = k$ and $\dim f(T) = 0$, $f(T)$ CH[32] and such that for *every* irreducible component $X_j \supset T_i$ of X with $\operatorname{codim}(T_i, X_j) = k$ we have $f(X_j) \not\subset H_\xi$. For a given T_i with $\operatorname{codim}(T_i, X) = k$ if there exists an X_j with $\operatorname{codim}(T_i, X_j) =$? [illegible, ask Grothendieck] and such that $\dim f(X_j) = 0$ then we will have $f(X_j) = f(T_i) \not\subset H_\xi$ and consequently ξ would not be exceptional relative to theT_i. If, on the other hand, for every $X_j \supset T_i$ such that $\operatorname{codim}(T_i, X_j) = k$, we have $f(X_j) > 0$ then for $\xi \in P^v - Z_0$, ξ is exceptional relative to T_i if and only if $f(T_i) \not\subset H_\xi$; the set of such ξ is (the trace over of $P - Z_0$ a hyperplane of

[32]Probably H_ξ [tr.]

P^v. This proves 8.4, and also proves the more precise result that the exceptional set is the union of a set of codimension ≥ 2 and of a *union of hyperplanes* determined in an evident way by the above proof.

(I am afraid that the writeup is quite floppy (or perhaps sloppy) [Tr] since I have reasoned geometrically all the time without saying so, by taking points over an algebraically closed field. Of course, the condition announced in 8.4 is indeed geometric so that we may suppose k algebraically closed and argue for k-rational points.) Using 8.4; 5.7.4 and the end of 8.2, we obtain:

Corollary 8.5. *Suppose that for all associated prime cycles R we have at most simply [illegible]*[33] *and suppose that F satisfies (S_k).*

In order that the (constructible) set of points of P^v such that ϕ_ξ is $F_{k(\xi)}$ regular and G_ξ is (S_k) should have a complement of codimension at least two it is necessary and sufficient to have the following: (⇔) for every integer $n \geq 0$ we denote byZ_n the set of $x \in T = \operatorname{supp} F$ such that the coprof$_x$ [illegible][34] *we see that for every irreducible component Z_{ni} of Z_n with codim$(Z_{ni}, T) = n + k + 1$ and* dim $f(Z_{ni} = 0$, *there exists an irreducible component T_j of T containing Z_{ni} such taht codim$(Z_{ni}, T_j) = n + k + 1$ and* dim $f(T_j) = 0$.

When f is quasifinite then for every closed subset R of [illegible, ask A.G.] we have dim $f(R) =$ dim R so that the criterion takes the following form: there does not exist an isolated point z in any one of the Z_n such that $\dim_z T (= \dim F_z)$ is equal to $n + k + 1$. When F is equidimensional of dimension d this condition is vacuous if $d \leq k$ (and indeed we knew it because in this case the [hypothesis] (S_k) on F is nothing else but the hypothesis Cohen-Macauley), and if $d \geq k + 1$ it means that the set $Z_{d-(k+1)}$ of points of T where the co-depth of F is $> d - (k + 1)$, i.e. true depth of$F \geq k + 1$ (even though, a priori, we only have true depth of $F \geq k$ as a consequence of the property (S_k) and $k \leq d$). If we no longer assume that F is equidimensional there remains that we may express the desired condition in the following simple way:

8.6. For every closed point $x \in \operatorname{supp} F$ such that dim $F_x \geq k_1$ we have $\operatorname{prof} F_x \geq k + 1$. The sufficiency is seen immediately by putting $Z = x$. The necessity is seen by noticing that for everyξ such that ϕ_ξ is $X_{k(\xi)}$-regular and$x \in Y_\xi$ we have dim $G_\xi x =$ dim $F_x - 1$, $\operatorname{prof} G_\xi x = \operatorname{prof} F)_x - 1$ so that x put by default the above condition we have $\operatorname{prof} G_x i x \geq k$ but dim $G_\xi x \geq k$ which shows that G_ξ does not satisfy condition (S_k) at x; but the set

[33] Illegible, ask A.G.
[34] Ask A.G.

of ξ such that $x \in Y_\xi$ is of codimension 1 (NB: I implicitly assumed that $k$is algebraically closed, the case to which we reduce immediately.) The preceding general criterion should be evident in the case 8.6.

We now study the points y of Y that are not smooth for Y_ξ relative to $k(\xi)$. We restrict ourselves to the case where $f\colon X \to P$ is *unramified* (practically, it will be an immersion) and where $X \to S$ is smooth. We do not necessarily assume that S is the spectrum of a field. Since f is unramified the canonical homomorphism $f^*(\Omega^1_{P/S}) \to \Omega^1_{X/S}$ is surjective and its kernel is a locally free module over X which we denote $\nu^v_{X'P}$; when if f is an immersion this is nothing else but the conormal module J/J^2 defined by the ideal J of X in P and we call it in every case the conormal module.

(a) $$0 \to \nu^v_{X/P} \to f^*(\Omega^1_{P/X}) \to \Omega^1_{X/S} \to 0$$

Let us observe that we have also over P an exact canonical sequence (which should appear as an example in paragraph 16 for example)

(b) $$0 \to \Omega^1_{P/S}(1) \to E_P \to O_P(1) \to 0$$

(i.e. $\Omega^1_{P/S}$ is canonically isomorphic to the kernel of the canonical homomorphism) $E_P(-1)$ $\to O_P$ deduced from $E_P \to O_P(1)$, to it we apply f^*:

(b[1]) $$0 \to f^*(\Omega^1_{P/S}(1) \to E_X \to O_X(1) \to 0$$

which gives an explicit description of $f^*(\Omega^1_{P/S})(1)$ over X and allows therefore to identify $\nu^v_{X/P}(1)$ with a submodule locally a direct factor of E_X or again th dual $\nu_{X/P}(-1)$ is canonically isomorphic to a quotient module of E^V_X. Consequently $P(\nu_{X/P}(-1)) = P(\nu_{X/P})$ can be canonically embedded into $P(E^V_X) = X x_S P^v = X^V_P$ as a projective sub-fibration over X therefore as a closed subscheme. The latter is necessarily contained in Y (from the fact that $\Omega_{X/P}(1)$ is contained in the kernel of $E_x \to O_X(1)$ [last two symbols illegible ask AG]

The underlying set of this prescheme is nothing else but the set of points of $Y = V(\phi)$ which are *singular zeros* (par. 16)[35] of the section ϕ of $\vartheta_{Xx^v_P}(1,1)$ relative to the base P^v, i.e. its points with values in the field k over P^v are the points x of $Y_k \subset X_k$ such tht ϕ_k vanishes to order at least two at x, i.e. such that Y_k is not smooth of relative dimension $r-1$ over k at x. The announced characterization of singular zeros [illegible, ask AG] the elements of a smooth subscheme $P(\nu_{X/P})$ of X^v_P gives immediately the following statement which deserves to appear as a preliminary proposition if $S = \operatorname{Spec} k$ and if H is a hyperplane

[35]See part II of these notes [Tr]

of P then $Y = Xx_PH$ is smooth over k of relative dimension $(d-1)$ at the point $x \in Y(k)$ (i.e. x is a non-singular zero, i.e. geometrically non-singular of the section ϕ of $O_X(1)$ defined by H) if and only if H does not contain the image by ϕ of the tangent space to X at x (relative to k) or as we say once more (if $f\colon X \to P$ is an immersion which allows us to identify X to a subscheme of P) if and only if H is not tangent to X at x. This follows trivially from the Jacobian criterion of smoothness or from the definition of a singular zero, once we make precise the sense of the statement, that is to say, that we make precise how a vector subspace of the tangent space to P at a point $a(= f(x))$ defines a linear subspace of P (in such a way that it makes sense to say that H does not contain the said vector subspace): of course this comes from the exact sequence (b) above which allows to define a one-to-one correspondence between the set of factor subspaces of the tangent space at a and the set of linear subspaces to P containing a. This correspondence anyway reduces to associating to a linear subvariety passing through a its tangent space at a considered as a subspace of the tangent space to P at a.

Such "sorites" grouped together with various "sorites" about linear subvarieties and about grassmanians ought to be given in one or two preliminary numbers or paragraphs of course announcing them over any base. In fact we can do better knowing that the prescheme Y^{sing} of singular zeros of ϕ relative to P^v defined in par. 16 is nothing else but $P(\nu^v_{X/P}$ and (since the latter is smooth over S of relative dimension $d + (r-d-1) = r-1$ (r being the relative dimension of P^v over S) we are under th favorable conditions studied in No. 16 or paragraph 16.[36] In order to verify them, let us notice that by definition Y^{sing} is nothing else but the sub-prescheme of Y of zeros of the section $\Psi = d\phi/Y$ of $\Omega_{X_{P^v}/P^v}(1,1) \otimes O_Y = \Omega^1_{X/S} \otimes O_Y(1,1)$ [illegible, ask AG][37]

We shall give another interpretation of this section from which the conclusion follows immediately. In order to do this let us consider the following diagram of exact sequences over X_{P^v} or more generally over any prescheme Z over X_{P^v}.

Diagram:

[36]Ask AG about this reference – just later part of these notes.
[37]Ask A.G.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & 0 & & G_{P^v/S}\otimes O_Z(0,-1) & & & & \\
 & & \uparrow & & & & & & \\
 & & \Omega^1_{X/Y}\otimes O_Z(1,0) & & \uparrow & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & \Omega^1_{P/S}\otimes O_Z(1,0) & \longrightarrow & E\otimes O_Z & \longrightarrow & O_Z(1,0) & \longrightarrow & 0 \\
 & & \uparrow & B & \uparrow & \propto & & & \\
 & & \nu^v_{X/P}\otimes O_Z(1,0) & & O_Z(0,-1) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

[Note to AG, the upper G is really an illegible letter P^v/S what is this?][38] where the first column is deduced from (a) by tensoring with $O_Z(1,0)$ the row is deduced from (b) by tensoring with O_Z and the column two is deduced from its transpose from the analogous sequence (b^v) relative to P^v (obtained by replacing E by E^v) and tensoring with O_Z. Fromt the very definition of Y, Z is over Y if and only if the composed morphism $\propto$ from the diagram is zero, i.e. if we can find a factorization $\beta\colon O_Z(0,01)\to\Omega^1_{P/S}\otimes O_Z(1,0)$. If this is the case we can consider its composition with $\Omega^1_{P/S}\otimes O_Z(1,0)\to\Omega^1_{X/Y}\otimes O_Z(1,1)$. I say that this is precisely the section ψ [Blass: check if this letter is OK][39] which we have introduced above (the verification ought to be essentially mechanical). It is zero if and only if Z is above lies over $V(\psi)$ (by the very definition of $V(\psi)$!) but this means also that β can be factored by $\nu^v_{x/P}\otimes O_Z(1,0)$, i.e. that the submodule $O_Z(0,-1)$ of $E\otimes O_Z$ is contained in the sub-module $\nu_{x/P}\otimes O_Z(1,0)$ which evidently signifies also that Z is over the sub-prescheme $P(\nu_{x/P}(1))$ of $P(E^v_X)$, achieving the proof that we have announced.

Just before this erudite exercise in syntax for which I have already had to sweat quite a bit we could remark that from every set theoretic point of view Y^{sing} is of dimension $r-1$ if $S=\operatorname{Spec} k$, whereas P^v is of dimension r so that the image of Y^{sing} in $P^v$is of codimension ≥ 1 which gives again 2.12 (it is well to note that the argument is not essentially distinct from the one used in 2.12). We note that most often this set is effectively of codimension one (compare below).

Consequently we cannot in general find the "linear pencils" of hyperplane sections all of which are smooth. However we shall see that we can often manage to find the pencils

[38] Ask A.G.

[39] Blass check this

formed by hyperplane sections not having any supersingular point due to the fact that in the most common cases the image of $Y^{\text{sup sing}}$ in P^v is of codimension two.

We shall first of all recall the essential points differential in nature of the situation studied here:

Theorem 8.7.

(a) *The sub-prescheme Y^{sing} (defined in No. or par. 16) in the present situation is nothing else but $P(\nu_{x/P})$ considered as a sub-scheme of Y as explained above.*

(b) *The underlying set of the prescheme $Y^{\text{sup sing}}$ (cf No. or par. 16) is nothing else but the set of ramification points of morphism of smooth preschemes over S of relative dimension $r-1$ and r (namely) $Y^{\text{sing}} = P(\nu_{x/P}) \to P^v$, i.e. in order for the latter morphism to be unramified at the point y (ref to the definition) it is necessary and sufficient that y should be geometrically an ordinary singular point for ϕ_ξ (ξ being the point of P^v that is the image of y).*

(c) *Let us assume $S = Spec(k)$ and that $y \in Y^{\text{sing}} = P(\nu)$ is a k-rational point, let [illegible][40] and ξ be its projections in $X(k)$ resp. $P^v(k)$ and let us consider the linear subvariety H^1 of P^v "image" of the tangent map of the closure of its [Fr. son image] image in P^v, given the induced reduced structure and let us consider the induced morphism $g\colon Y^{\text{sing}} \to T$ (a dominant morphism of integral preschemes). The conditions (i) and (ii) (bis) are equivalent:*

(i) *The morphism g is generically étale (i.e. étale at least one point or what is the same is étae = unramified at the generic point of Y^{sing})*

(i bis) *The field extension L/K defined by g is finite and separable.*

(i ter) *The morphism g is birational, i.e. the extension L/K is the trivial extension.*

(ii) *$Y^{\text{sing}} \neq Y^{\text{sup sing}}$ (set theoretically speaking let us say*

(ii bis) *There exists an $x \in X(k)$ and a tangent hyperplane H to X_k at x which is not osculating at x by which we understand precisely that x is not supersingular for the section of $O_{Xk}(1)$ tht defines $H\ldots$).*

These conditions imply that $Y^{\text{sup sing}} \neq \phi$ [Fr. illegible, ask A.G.] $\dim Y^{\text{sup sing}} \leq r-2$ so that the image of $Y^{\text{sup sing}}$ in P^v has a codimension ≥ 2, and they imply also

(iii) $\dim T = r-1$, *i.e. T is of codimension one in P^v.*

Proof. The equivalence of (i) and (i bis) is trivial its equivalence with (ii) is a trivial consequence of 8.7 b), finally the equivalence of (ii) and of (ii bis) is practically the definition of $H^{\text{sup sing}}$. Evidently (i ter) $\Rightarrow$ (i) it remains to prove that (i) $\to$ (i er). We may evidently

[40] Ask A.G.

suppose that K is algebraically closed and we are reduced to prove (taking into account the hypothesis (i)) that there exists an open set $U \neq \phi$ such that $\xi \in U(K)$ implies that there exists exactly one point of $Y^{\rm sing}(K)$ over ξ. This will follow from 8.7 c) which implies more precisely.

Corollary 8.9. *Suppose that condition (i) of 8.8 is satisfied and let U be the open subset of T of the points where T is smooth over k. Then $U \neq 0$, $Y^{\rm sing}/U \Rightarrow U$ is an open immersion a fortiori $Y^{\rm sing}/U$ does not contain the points of $Y^{\rm sup\ sing}$. If X is proper over K, then $g\colon Y^{\rm sing} \to T$ is surjective thus $Y^{\rm sing}/U \Rightarrow U$ is proper over K so that $g\colon Y^{\rm sing} \to T$ is surjective therefore $Y^{\rm sing}/U \to U$ is an isomorphism and U is the biggest open set of T having the latter property.*

First of all since g is dominating and generically étale it is generically étale so we can find at least one non-empty open subset V of T such that $Y^{\rm sing} \mid V \Rightarrow V$ is étale and surjective which implies that V is smooth over K. If then $\xi \in V(K)$ and if $y$is a point of $Y^{\rm sing}(k)$ over ξ then with the notations of 8.7 c) the space H' is nothing else but the tangent space to T and ξ, and as we observed here this implies that the point x of $X(k)$, the projection of y is determined as the orthogonal point to H' thus it is uniquely determined thus since $Y^{\rm sing} \subset XxP^v$ is uniquely determined.

This proves already that g is birational (being generically étale and generically radical). On the other hand the morphism ψ (whose definition in its form is evident) which associates to every $\xi \in U(K)'$ the unique point $x = \psi(\xi) \in P$ orthogonal to the tangent space to U at ξ, coinciding over V with the composition $V \to Y^{\rm sing} \mid V \to X$, where the second arrow is the projection; therefore setting $h = (\psi, {\rm id})\colon U \to PxT$ (illegible)[41] $g_1 = g \mid g^{-1}(U)\colon g^{-1}(U) \to U$ the composition $hg_1\colon g^{-1}(U) \to PxT$ is nothing else but the canonical inclusion, this being so for its restriction to $g^{-1}(V) \xrightarrow{\sim} V$. It follows that h factors thrugh the scheme theoretic closure $\overline{Y^1}$ of Y^1 in PxT thus that the inverse image of Y^1 (which is open in the above closure) by h is an open subset of U, let us call it U^1. Because of $hg_1 =$ inclusion we see immediately that g^1 induces an isomorphism $g^{-1}(u) \xrightarrow{\sim} W$ is an isomorphism it follows that W is smooth since Y^1 is smooth, thus $W \subset U$. This proves 8.9

The final assertions of 8.8 $Y^{\rm sup\ sing} = \phi$ or $\dim Y^{\rm sup\ sing} = r - 2$ and $\dim T = r - 1$ are trivial: the first one follows the fact that $Y^{\rm sing}$ is irreducible of dim r and from the fact that $Y^{\rm sing}$ or $Y^{\rm sup\ sing}$ [illegible][42] is defined by the vanishing of a section $D$of an invertible module; the second from the fact that L being finite over K we have deg tr $L/k =$

[41] Ask A.G.
[42] Ask A.G.

deg tr K/k, i.e. $\dim T = \dim Y^{\text{sing}} = r-1$.

Remark 8.10. As we remarked in 8.9 with the notations of the corollary we have $g^{-1}(U) \subset Y^{\text{sing}} - Y^{\text{sup sing}}$ but we notice that even if X is closed in P this inclusion is not necessarily an equality, in other words (noting that $g^{-1}(U)$ is nothing else but the set of points where g is étale, so that $Y^{\text{sing}} - Y^{\text{sup sing}}$is the set of points where g is unramified but not étale (which implies in addition that $g(y)'$ is a point that is not geometrically normal and even not geometrically unibranch of T). In geometric terms this corresponds to the following phenomenon; we may have a tangent non-osculating hyperplane for X at a point $x \in X(k)$ such that there exists another point $x^1 \in X(k)$ at which the same hyperplane is tangent at x [or x^1 illegible].[43] Indeed there are obvious examples with P of dim two,X a non-singular curve of degree ≥ 4, in any characteristic. [Note here: the "double tangents" of X correspond to the double points of the "dual curve."]

Corollary 8.11. *Let us assume that k has characteristic zero. Then*

(a) The image of $Y^{\text{sup sing}}$ in P^v is of codimension ≥ 2.

(b) The condition (iii) of 8.8 is equivalent to other conditions, i.e. the negation of the other conditions, let us assume that $Y^{\text{sing}} = Y^{\text{sup sing}}$ means also that the image of Y^{sing} [or $Y^{\text{sup sing}}$ illegible] in P^v is of codimension ≥ 2.

Evidently, the assertion (b) implies (a) taking into account 8.8. But by dimension theory, (iii) means that L/K is a finite extension (we could put it in the form ((iii) bis) in 8.8 and in characteristic zero L is always separable over K hence the condition (i bis) of 8.8.

Remark 8.12. Geometrically the assertion (a) means essentially that for a sufficiently general linear pencil of hyperplane sections every member of the pencil is smooth or has for geometric points singular points ordinary double points (and in fact as one sees immediately it can be said in statement (a) consequently in a form a little more precise – we have at most *one* such singular geometic point). The assertion (b) means essentially that if $Y^{\text{sing}} = Y^{\text{sup } sing}$ (which can be expressed analytically by the vanishing of a certain section D of an invertible module $\omega_{X/k}^{\otimes 2} \otimes O_{Y'}$ $(1,1)$ over Y^1), then for a sufficiently general linear pencil of hyperplane sections all the members of the pencil are smooth. This second situation (whether or not we are in characteristic zero) should entirely be considered as exceptional. [The variety L in $1 \ldots T = L$ illegible handwriting on topof the page][44] In the classical language it is expressed, if there is no error, by saying that X is ruled for the

[43] illegile
[44] illegible

projective immersion considered [and if we so please] we have here all that we need due to 8.5 and its corollaries to make explicit and justify such a terminology in case if you feel inspired to make connection with [la taupe Fr]. For example if dim $X = 1$ this implies that X is a straight line [illegible][45] $x \in X(k)$ so T conains[46]. (b) If the characteristic is $p > 0$, we should normally give examples (with dim $P = 2$, X a non-singular algebraic curve) where the conditions of 8.6 are not satisfied, i.e. $Y^{\text{sing}} = Y^{\text{sup sing}}$ and where nevertheless dim $T = r - 1$, i.e. examples where L/K is a finite inseprable extension. I am too lazy to construct the examples but I do not doubt that such examples exist.[47]

In (a) make a footnote to the following No. or paragraph where we prove that if the exceptional 'ruled' case arises then by a trivial modification of the projective immersion we find ourselves again in the "general" situation of 8.8.

the part of the present section from 8.6 to here could without a doubt be made into a separate section of a differential character, whereas the beginning of our No. with the one that follows should be merged together into a No. about the dimension of exceptional [hyperplanes???][48] I only use the fact that Y^{sing} has dimension $(r-1)$.

Proposition 8.13. *We always assume that $f\colon X \to P$ is unramified and that X has no isolated points. We assume that X satisfies (R_k) geometrically.*

Let Z_k be the part of P^v complement of the set of $\xi \in P^v$ such that ϕ_ξ should be $X_{k(\xi)}$-regular and Y_ξ satisfies the geometric condition E_k then:

- *a) In order for Z_{k-1} to be of codimension ≥ 2 in P^v it suffices that every irreducible component x'_i of X' of dimension $\leq k$ should be ruled for f.*
- *b) In order to have Z_k of codimension ≥ 2 in P^v it suffices that every irreducible component X_i of dimension $\leq k-1$ (Ask A.G. illegible) should be made up of smooth points of X and should be ruled.*

Indeed for every ξ geometrically singular the set of Y_ξ (NB: We restrict ourselves to ξ such that $\phi_x i$ is $X_{k(\xi)}$ regular which is harmless because of 8.2) and is the union of $\text{sing}(Y'_\xi)$ and of the inverse image T_ξ of T in Y_ξ so thta the codimension of this singular set in Y_ξ is equal to the infimum of the codim $(\text{sing}(Y'), Y'_\xi)$ and of the codim (T_ξ, Y_ξ). Let us restict ourselves to ξ such that $\text{sing}(Y'_\xi)$ is finite (which is harmless, this leads to place ourselves in the complement of a set of codimension ≥ 2). The singular geometic points of Y'_ξ are therefore isolated. The conclusion follows easily from this and from 8.4.

[45] illegible
[46] illegible
[47] Do itBlass
[48] Ask A.G.

Combining 8.13, 8.5 and 8.6, we find in the usual manner

Corollary 8.14. *We suppose that $f\colon X \to P$ is unramified and that X has no isolated points [illegible] n.*[49]

a) *Suppose X is separable over k. In order that the set of $\xi \in P^v$ such that $\phi_x i$is $X_{k(\xi)}$-regular and Y_ξ is separable, should have a complement of codimension at least two it is necessary and sufficient that every irreducible component X_i of dimension one of X should be formed from smooth points of X and should be ruled relative to f and it that for every closed point x of X such that $\dim_x X \geq 2$ we have prof$x$ $X \geq 2\,\mathrm{prof}_x X$, (conditions that are automatically satisfied if X is geometrically normal and if all of its irreducible components are of $\dim \geq 2$).*

b) *Let us assume that X is geometrically normal, in order that the set of $\xi \in P^v$ such that ϕ_ξ is $X_{k(\xi)}$ regular and Y_ξ is geometrically normal should have a complement of codimension at least two it is necessary and sufficient that every irreducible component X_i of X of dimension ≤ 2 should be formed of smooth points of X and that it should be ruled relative to f and that in addition for every closed point x of X such that $\dim_x X \geq 3$ we have prof$_x X \geq 3$.*

Remark 8.15. In 8.6, 8.13, and 8.14 we make for X the hypothesis (S_k) (resp. (R_k) respectively: separable, respectively geometrically normal) that we wish to recover as a conclusion for the hyperplane sections except perhaps for ξ from an exceptional set of codimension at least two.

This does not restrict the generality; to tell the truth, it would have been better to get rid of this preliminary hypothesis, since we see immediately with the help of results of par. 3.4 and 5.12 that if X does not satisfy the hypothesis in question, then (by par. 5) if there exists a closed point x where the hypothesis fails then for everyξ such that ϕ_ξ is $X_{k(\xi)}$ regular condition that only eliminate a set of codimension (illegible) and such that $x \in Y_\xi$ (condition that describes a set of exact codimension one), Y_ξ does not satisfy the said hypothesis at x, the exceptional set $Z \subset P^v$ is of codimension one and not two. (I may have somewhat exaggerated the case E_k where we still need some condition, (S_1) and perhaps of equidimensionality perhaps ...)

Remorse In 8.13 and 8.14 it suffices to suppose that $f\colon X \to P$ is unramified at the smooth points of X; for the sufficiency part it suffices only that they should be unramified over an open subset X' of X where complement has codimension $\geq k+1$.

[49] Ask A.G.

Proposition 8.16. *Let us suppose $f\colon X \to P$ unramified on an open subset complementary to a set of codimension at least two, X geometrically normal and of depth at least three at its closed points, finally X geometrically integral and proper over k. Then the set of $\xi \in P^v$ such that Y_ξ is geometrically normal and geometrically integral of dimension equal to* $\dim X - 1$ *(is constructible and) has a complement of codimension at least two.*

Indeed by 8.14 b) such is the case for the property "Y_ξ is geometrically normal of dimension $\dim X - 1$" (the dimensional property expresses that ϕ_ξ is $X_{k(\xi)}$-regular.)

Therefore, by 6.1 all the Y_ξ are geometrically connected. Since Y_ξ is geometrically normal it is geometrically integral if and only if it is geometrically connected, which gives the proof.

Remarks 8.17.

a) The hypothesis of 8.16 implies that $\dim X \geq 3$. It is possible that [de's que. Fr.] X is geometrically irreducible and that $\dim f(X) \geq 3$ (without the hypothesis of normality and of non-ramification) the set of ξ such that Y_ξ is geometrically irreducible has a complenent of codimension at least two. We can prove in every case that it does not contain a hyperplane (see below).

b) The conclusion of 8.16 is false if we leave out the assumption that $\mathrm{prof}_x X \geq 3$ for x closed, for example it is false for a non-singular quadric X in P^3 [illegible][50] the hyperplane sections are reducible (in fact formed by pairs of concurrent lines) and they form therefore a two dimensional family thus of codimension one in P (indeed the dual of the quadric is a quadric in the dual space relative to the dual form...)

In the case of a non-singular surface in a projective space this situation however should be considered exceptional of the following section (or No.). Let us suppose [illegible] integral proper over k and f an immersion. Then it follows from 6.1 and 8.8 and 8.14 that if $Y^{\mathrm{sing}} \to P^v$ is not generically finite and inseparable one of the (???) $\xi \in P^v$ such that Y_ξ is separable over $k(\xi)$ with at most two irreducible [illegible] a complement of codim at least two.

We shall now examine more precisely the case of surfaces (the case of curves does not arise evidently, from the point of view of irreducibility of hyperplane sections).
(NB: I noticed with fright that the quadric is not entitled to be called ruled in the sense that I have been using the word ruled. This is in disagreement with our fathers and it would be necessary to invent a more adequate word for the notion used here.)

[50] Ask A.G.

Proposition 8.18. *Let us suppose that k is algebraically closed, X is integral (respectively integral and normal) of dimension ≥ 2 and proper over k, let T be a closed finite subset of X such that $X - T$ is smooth and let $f/X - T$ be unramified. In order that the set of $\xi \in P^v$ such that Y_ξ should be geometrically irreducible (respectively geometrically integral) of dimension 1 should have a complement of codimension ≥ 2 it is necessary and sufficient that the following conditions should be satisfied:*

a) For every $x \in T$ there exists a hyperplane section Y_ξ ($\xi \in P(k)$) passing through x of dimension $d-1$ and which is irreducible,

b) $X' = X - T$ is "ruled" (sic) for f or there exists a hyperplane section Y'_ξ ($\xi \in P^v(k)$) of X' which is of dimension $d-1$ (non???) singular and irreducible.[51]

Let us first assume that X is geometrically normal. We have already seen then by 8.14 a) that we can find a closed *subset* Z' of P of codim ≥ 2 such that $\xi \in P - Z'$ implies that Y_ξ is separable over $k(\xi)$ and of dimension $d-1$ for such a ξ, it amounts to the same that Y_ξ should be geometrically irreducible or geometrically integral, and the two problems [(respé et non respé) Fr. p. 50] (?) considered in 8.18 are therefore equivalent. On the other hand, by 5.6, the set U of $\xi \in P$ such that Y_ξ is geometrically integral of dim $d-1$ (the dimension hypothesis stating that ϕ_ξ is $X_{k(\xi)}$ regular) is *open.* We will exhibit a non-empty open *evident* subset $P - Z$ contained in U and taking for Z the *union of* $g(Y'^{\text{sing}})$ and of the hyperplanes H_x of P^v defined by the $f(x)$, $X \in T$. For $\xi \in P^v - Z$, Y_ξ is smooth of dimension $(d-1)$ and since it is geometrically connected by 6.1 it is geometrically integral. We have to, therefore, express (explain) (prove) that every irreducible component of codimension one of Z meets the open set U. But these irreducible components are the H_x [they are repeated possibly, but it is not essential] and also $g(Y'^{\text{sing}})$ when the latter are indeed of codimension one, i.e. X' "not ruled" for f (Nota Bene: we use the irreducibility of Y'^{sing}. On the other hand, in order that this latter set should meet the open set U it is necessary and sufficient that $g(\overline{Y'^{\text{sing}}})$ which contains an open and dense set) should meet U. This proves 8.18 in this case. If we do not suppose that X is normal, we apply the previous result to the normalization of X the reasoning is immediate and I do not give the details here. N.B. In the *case* [respé] 8.18 is contained in 8.16 more precisely except in the case $d = 2$. It is for the case *[non-respé] that it may be better not to require* $d = 2$ and not only $d \geq 2 \ldots$.

It remains to explain (make explicit) the conditions a) and b) of 1.18. This leads us to examine in a general way the following situation. We suppose that X is geometrically irreducible over k and we (give ourselves) consider a linear subariety L of P (corresponding

[51] illegible, ask A.G.

to the question of studying the hyperplane sections of X, passing through a given point x or tangent to X at a given smooth point), formed therefore by the hyperplane containing a linear subvariety L of P (resp. a point, or the image of a tangent space to X at a smooth point in the two cases considered) and we ask the question [de ravoi] if for the generic point of L (therefore for all the points of a non-empty open subset of L) Y is geometrically irreducible of dim $= \dim X - 1$. This is a variant of Bertini's theorem, which [j(devrait figurer) 51] must appear in No. 3, and is treated by exactly the same method, [(ou, si on veut, s'y ramène) Fr 51]. The dimension question is simply stated for $f(X) \notin L^{-}0$, i.e. if $X' = f^{-1}(P - L^0)$ is a dense open subset of X. Let Q be the projective space of hyperplanes passing through L^0 (N.B. if L^0 is defined by a vector subspace F^0 of E we have $Q = P(F^0)$ and we consider the canonical morphism (deduced from $F^- \to E$, cf. Chap II).

$$u \colon P - L^0 \to Q$$

and we consider

$$g = uf' \colon f^{-1}(P - L) = X' \to Q$$

so that $L \simeq \overset{V}{Q}$ and the family of X'_ξ ($\xi \in L$) is nothing else than the family of hyperplane sections relative to the morphism g. On the other hand, we see immediately that for every $\xi \in L$, "*general*" X' is dense in X, so that X' is geometrically irreducible if and only if X is such. This assumed, the theorem of Bertini-Zariski shows us that we have the wanted conclusion of irreducibility provided that $\dim g(X') \geq 2$. (To tell the truth, one could give a converse to 3.1 as follows: If X is geometrically irreducible Y is geometrically irreducible if and only if either $\dim f(X) = 2$ *or* $\dim f(X) = 1$ and $f(X)$ is contained in a straight line D defined over k and the generic fiber of $X \to D$ is geometrically irreducible.) This also allows us in the present version with L to have a necessary and sufficient condition of geometric irreducibility of Y_ξ, ξ generic in L.

From the [(cunutesque?) Fr] point of view and in terms of field theory we can express the condition in terms of transcendence degree in the following fashion. We choose a "hyperplane at infinity" containing neither L^0 nor X and we place ourselves in its complement, i.e. over a scheme of affine type essentially. We choose a basis of the space of linear forms vanishing on L, let it be $T_1, \ldots, T_p$ ($p = \operatorname{codim}(L^0, P)$) and we consider their inverse images $t_1, \ldots, t_p$ in the field of fractions K of X (X assumed integral). At least one of the t_i, let us say t_1 is $\neq 0$. Let us consider therefore $a_1 = t_2/t_1, \ldots, a_{p-1} = tp/t_1$ then $\dim g(X')$ is nothing else but the transcendence degree of $K(a_1, \ldots a_{p-1}) \subset K$ over k. Therefore if the transcendence degree if ≥ 2 we are o.k. If it is one then we must require that over k, $f(X)$ is contained in a linear subvariety of P containing L^0 and of dimension at most one and that the *generic* fiber of $g: X' \to g(X')$ should be geometrically irreducible.

Let us suppose that L^0 is of dimenison g, so that the fibers of $u: P - L^0 \to Q$ are of dimension $q+1$ so that those of g are dim $\leq q+1$, and consequently we have dim $g(X') \geq$ dim $f(X) - (q+1)$ so that the dimension condition for $g(X')$ is verified in view of the fact that dim $f(X) \geq q+3$. If $q = 0$ we find the fact indicated in 8.17 a). Returning to conditions of 8.18 we see that condition a) relative to an $x \in T$ is satisfied provided x is not "conical at x relative to f" in an obvious sense. Maybe it will be better to introduce these latest *Bertinisque* developments in the next section... change of projective immersion.

§9 Change of Projective Embedding

9.1. For every integer $n > 0$ let $P(n) = P(\mathrm{Sym}^n(e))$, we have an evident immersion $u_n: P \to P(n)$, since $\mathcal{O}(n)$ is generated by its sections over every open affine of S and that $p_*(0_p(n))\mathrm{Sym}^n(E)$[52] where $p: P \to S$ is the projection. If $f: X \to P$ is an unramified morphism (resp. an immersion) it is the same with $u_n f: X \to P(n)$. There is sometimes an advantage in the study of X in replacing f by $u_n f$ in order to avoid a very special behavior and sometimes embarrassing to f in certain respects. (An example of such peculiarity is the one indicated (sic) in 8.12 b), where $Y^{\mathrm{sing}} \to P^v$ has an image of dimension $r-1$ but gives rise to an inseparable extension of fields. Another one is that given by the quadric surfaces in P^3 to know that all the singular hyperplane sections are geometrically reducible. (in spite of the fact that X is geometrically irreducible.))

Proposition 9.2. *We suppose $S = Spec(k)$, X smooth over k, and $f: X \to P$ unramified. Let $n \geq 2$ and let us consider $f_n = u_n f$. Then $f_n: X \to P(n)$ satisfies the equivalent conditions of 8.8, in particular for $\xi \in P(n)$ in the complement of a set of codimension ≥ 2, the corresponding hyperplane section Y_ξ is smooth or (admits) only a finite number of non-smooth points which are geometrically ordinary singularities. If f is an immersion there is at most one such singular point and it is rational over $k(\xi)$.*

N.B. One would have to announce 8.8 in a manner such as not to exclude the case where f is not an immersion. The verificatioin is essentially trivial under the condition (ii bis) of 8.8. Without a doubt we should make explicit in 9.1 that the hyperplane sections of X relative to f_n are nothing else but the "sections" of X by hypersurfaces of degree n in place of hyperplanes.

[52] Illegible

Proposition 9.3. *Suppose that X is geometrically irreducible and that* $\dim f(X) \geq 2$, *let $x \in X(k)$ let $n \geq 2$.*

a) *Let us consider the linear family of hyperplane sections of X relative to $f_n = u_n f$ which pass through x, its generic element defines a $Y_\xi^{(n)}$ which is geometrically irreducible.*

b) *Let L be a linear subvariety of P passing through $f(x)$ and not containing $f(X)$. Let us consider the linear familyof hyperplane sections of X relative to f_n defined by the hyperplanes of $P(n)$ "tangent to L at x" (i.e. defined by the n-forms over P which over L are zero of order at least two at x), its generic member is a $Y_\xi^{(n)}$ which is geometrically irreducible.*

c) *Let us suppose that X is smooth at x and that $n \geq 3$ where $f(X)$ is not contained in a plane defined over k. Let us consider the family of hyperplane sections $Y_\xi^{(n)}$ of X relative to f_n which are "tangent to X at x". Then the generic member of the latter defines a $Y_\xi^{(n)}$ that is geometrically irreducible.*

The proof is essentially trivial in terms of the criteria of the end of the previous section. Taking an affine model of P containing $f(X)$ we are reduced a) to finding three polynomials in the coordiantes $T_1, T_2, \ldots, T_r$ of degree ≤ 2, let them be P, Q, and R such that $Q(t)/P(t)$ and $R(t)/P(t)$ are algebraically independent over k in K (where K is the function field of X and $t = (t_1, \ldots, t_r)$ is the system of elements of K defined by the T_i); in b) we require also that P, Q, and R should vanish to order two at least on L which we can in addition suppose to be defined by the equations $T_1, \ldots, T_s = 0$; finally, in c) it is the same but L is the image of the tangent space ot X at x and we allow possibly to take P, Q, and R of degree 3, i.e. a little more away. The hypothesis that $\dim f(X) \geq 2$ signifies that the transcendence degree of $K(t_1, t_2, \ldots, t_r)$ over k is ≥ 2, i.e. we can find t_1, t_2 let us say algebraically independent. In a) we take therefore $P = T_1$, $Q = T_1^2$, $R = T_1 T_2$, in b) we (analogously) do the same noting that we may there choose t_1 leading to T_1 zero over L due to the fact that $f(x) \not\subset L^*$ (which implies that there exists an index i between one and s such that $t_s \neq 0$, so that t_s is not a constant (since t_s is zero at x) therefore t_s is not algebraic over k[53] (N.B. we may suppose k algebraically closed). The case c) follows from b) except in the case where $f(X)$ is contained in the image, L, by f of the tangent space to X at x. [If dim $X = 2$ this case is effectively exceptional (the trace of a quadric surface tangent to a plane on that plane is in general formed by two intersection lines and is therefore not irreducible)]. But to treat that case, in the forms P, Q, and R made explicit above we may replace evidently X by L itself, where the solution is trivial. (If

[53] [Tr] added by translator

dim $L = 2$ take $P = T_r^2$, $Q = T_r^3$, $R = T_r^2 T_{r-1}$ and not that $Q/P = T_r$ and $R/P = T_{r-1}$ are linear forms independent over L, therefore algebraically independent. If dim $L \geq 3$ then T_{r-2}, T_{r-1}, T_r are linearly independent over L and we take

$$P = T_r^2, \; Q = T_{r-1}^2, \; R = T_{r-2}^2$$

Conjugating with 8.18 we find a Corollary 9.4. ([Tr] to be stated)

Finally we must combine the latter with 9.2 in order to find a recapitulating theorem in the "excellent case."

Theorem 9.5 (illegible page 52 or 55). [54]

(If X is smooth and proper and geometrically integral over k and $f\colon X \to P$ is unramified X of dimension ≥ 2 (Ask Grothendieck) by considering the result ??? when $x \to p$ is an immersion (variety of singular points of Y_ξ.

§10 Pencils of hyperplane sections and fibrations of blown up varieties

10.1. Let Z be the P-exceptional set in P^v relative to a constructible property P such that Z is a constructible subset of P^v. Let us suppose $S = \operatorname{Spec}(k)$. We will see (cf. No. 12 where we catch up with things which should have come in without a doubt in (previous Nos.) that in order to have $\operatorname{codim}(Z, P^v) \geq 2$ it is necessary and sufficient that "every sufficiently general line" L in P^v should not meet Z or also again (or even) Z and it suffices that there should exist one (a single) L in P not meeting Z *(should be Z; AG's error P.B.).*[55] If k is infinite it is necessary and sufficient that there should exist a single straight line L in P that does not meet $\bar{z}$. We call a linear pencil of hyperplane sections of 'X" defined by the straight line L in P the L-prescheme Y_L (definition valid for any S). Then the previous reflections together with results of Nos. 8 and 9 give us criteria for the existence of such pencils having the fibers Y_ξ ($\xi \in L$) all satisfying the property P first of all in the case where S is an infinite base field. Taking into account 8.2, if for every associated prime cycle on X we have dim $f(T) > 0$ then we can (by taking the property $P' = P+$ condition of regularity for ϕ_ξ) require that the pencil Y_L should be flat over L. In the case where S is arbitrary we can again, proceeding by the procedure of 7.1, construct such a pencil over an open neighborhood of a given point s of S in view of the fact that $k(s)$ is infinite and we should know that Z is closed (which is assured in diverse various misce laneour cases by the results of Par. 5 and the assumption that $X \to S$ is proper). To do it right it would be convenient after general explaination of this type

[54] Ask A.G., I do not follow [Tr]
[55] Ask A.G.

to give recapitulating statements where we effectively apply the preceding results for a certain number of properties of this nature (and also comprising module properties). As a minimum in this sense we must give here the reformulation of 9.5 in terms of linear pencils – a fact constantly used in geometric applications.

10.2. By a polarity, to a straight line L in P^v there corresponds a linear subvariety L^0 of codim 2 in P (S arbitrary). Let us put $T = x_P L^0$. Another way to describe T is as follows; L is defined by a locally free quotient of rank 2 of E^v or what is the same by a submodule, locally a direct factor F of E everywhere of rank two. Let us consider the composed homomorphism

$$F_X \longrightarrow E_X \longrightarrow O_X(1),$$

then T is nothing else but the scheme of zeroes of this composed homomorphism or what is the same it is defined by the ideal J, image of the corresponding homomorphism (obtained by twisting by $O_X(-1)$)

$$: F_X(-1) \longrightarrow O_X.$$

Let us suppose that this homomorphism is regular which means that if we write down locally a totally ordered basis of $F_X(-1)$ then its image in O_X forms an O_X-regular sequence, a condition that does not depend on the basis chosen and that can be announced intrinsically also by saying that $F_X(-1) \otimes O_X/J \to J/J^2$ is an isomorphism and tht $V(J) = T \to X$ is a regular immersion [NB: we should somewhere reveal the general situation with a homomorphism $G \to O_X$, G locally free over the prescheme X, for example in the section about regular immersions] we have then

Theorem 10.2. *With the above hypothesis the linear pencil Y_L with the canonical projection$Y_L \to X$ is X isomorphic in a unique fashion to the blow-up of the prescheme X with center T.*

To understand the meaning of this theorem it is convenient to notice at the beginning of the section orno that if $S = \mathrm{Spec}(k)$ then for a 'sufficiently general' straight line L in P^v the condition of regularity is verified (cf. *catching up indicated in No. 12* namely for 5.3.) In what follows in the construction of "good" linear pencils indicated or anticipated at at the beginning of the present No., we could require that the described pencil should satisfy the said condition (which is a condition of the same type but different from the one that consists in requiring that for every $\xi \in L$, ϕ_ξ should be $X_{k(\xi)}$-regular). We should include the condition in question in the proposed recapitulating statements.

On the other hand, practically 10.2 is used only in the situation of 9.5, which makes it desirable not to announce the reformulation of 9.5 in terms of pencils, until after 10.2, in order to be able to include in the statement in question also the isomorphism of the pencil with a blow up (i.e. to give a description of the situation permitting a suitable reference). We obtain thus a way for every projective smooth geometrically connected X of dim ≥ 2 over an infinite field k, to find a closed smooth subscheme non-empty of codimension two at every one of its points such that the blown up scheme admits a fibration over P^1, with all the fibers are geometrically integral and such that all the fibers are smooth except at most a finite number, the latter having at most a geometrically singular point and such a point being rational over k and geometrically an ordinary singularity.

This explains the importance of a deep study (just started at the present time) of such fibrations with singular fibers to reduce in a certain measure (to some extent the study of projective smooth varieties of dimension d to those of (families depending on one parameter) or projective varieties of dimension $(d-1)$ that may have ordinary singularities.

The statement 10.2 is a more or less immediate consequence of the following which is completely independent of the story of hyperplane sections and would be without a doubt better in its place in an extra paragraph "regular immersions."[56]

Proposition 10.3 is crossed out. Ask AG if that is his intention.

Proposition 10.3. *Let X be a prescheme, G a quasi-coherent module over X and $u: G \to \mathcal{O}$ a homomorphism, $J = u(G)$, $T = V(J)$. Let X be deduced from X by blowing up T. Let us consider on the other hand $p = p(G)$, the canonical homomorphism $G_p \to \mathcal{O}(1)$ and its kernel H (such that we have the exact sequence $0 \to H \to G_p \to \mathcal{O}(1) \to 0$) the homomorphism $u_p: G_p \to \mathcal{O}$ and the quadi-coherent ideal $K = u_p(H) \to \mathcal{O}$. Then X is canonically isomorphic to a closed subscheme of $V(K)$. If G is locally free and u is "regular" then the above isomorphism is an isomorphism of X with $V(K)$ itself in this case H is locally free over p and $H \to \vartheta_p$ 9whose prescheme of ??? is X) is also regular.*

The first statement is almost trivial. The second one is an exercise which does not cause any difficulty (I have not done it in detail thinking that you can deduce it just as well as I).

If in 10.2, $S = \operatorname{Spec} k$ and X is of dimension ≥ 1 then the assumption of regularity made is equivalent to $T = \phi$ so that $Y_L \to X$ is an isomorphism. We find therefore by conjugating with 9.2:

Corollary 10.4 (of 10.2). *Let X be a smooth curve geometrically connected in a projective space P over an infinite field k and let $n \geq 2$. Then there exists a linear pencil of n-forms over P defining a morphism $X \to P^1$ having the following property: the morphism*

[56] Ask A.G.

is generically étale of degree d and for eery geometric point s of P^1, X_s is étale over the algebrically closed field $k' = k(s)$ or it is k' isomoprhic to the sum of $d-2$ schemes Spec k and the scheme $I'_k = \operatorname{Spec} k'[t]/(t^2)$. In the language of the forefathers: there is at most one point of ramification and it is "quadratic."

§11 Grassmanians

Since we will now use linear subvarieties of P not only of reltive dimension 0 and$n-1$ it is clear that we shall need some notations about grassmanians and some [‘*sorites*’][57] (facts) of the nature of ‘elementary geometry’ about the constructions concerning linear varieties which should all come at the beginning of the paragraph. In addition *one takes in practice sometimes* any linear sections and not only hyperplane sections and it is proper to review (revisit) in this enlarged spirit all the previous Nos. sections.

Let E be a quasi-coherent module over the prescheme S, and let n be an integer > 0. Let us consider the functor $(\mathrm{Sch})^0/S \to (\mathrm{Ens})$ defined by $\mathrm{Grass}_n(E)\ (S')$ = quotient modules, locally free and of rank n of E'_s.

This functor is representable and the prescheme over S which represents it will also be denoted $\mathrm{Grass}_n(E)$. To prove the representability consider the natural homomorphism of functors

$$\mathrm{Grass}_n(E) \longrightarrow \mathrm{Grass}_1(\Lambda^n E) = P(\Lambda^n E)$$

defined by associating with every locally free quotient of rank n, G of E'_s the locally free module of rank one G considered as a quotient of E'_s. We prove as in *Seminaire Cartan*[58] that this morphism if "representable by a closed immersion" (*for closed immersions*) such that $\mathrm{Grass}_n(E)$ appears as a closed subscheme of $P(\Lambda^n E)$; in particular it is separated over S and quasi-compact over S and if E is of finite type it is projective over S. If E is of finite presentation then that is also the case for $\mathrm{Grass}_n(E)$: indeed we may suppose that S is affine$S = \mathrm{Spec}(A)$ so that E comes from a module of finite type over a subring of finite type of A – since the formation of $\mathrm{Grass}_n(E)$ is evidently compatible with base change over S.

Since E is locally free therefore $\mathrm{Grass}_n(E)$ is smooth over S with geometrically connected fibers. This comes from a more precise fact: If E is free of rank r then $\mathrm{Grass}_n(E)$? may be recovered from $\binom{r}{n}$ open subsets each one of which is S isomorphic to affine space of relative dimension $n(r-n)$ over S. This decomposition corresponds to the choice, thanks to the base of E to $\binom{r}{n}$ decompositions of E by exact sequences $(s)0 \to E \to E \to E'' \to 0$

[57] What is best translation of this word?
[58] Make reerence more precise (Tr)

with E' locally free of rank n. Such an exact sequence allows us to define a sub-functor $\mathrm{Grass}_n(s)$ of $\mathrm{Grass}_n(E)$ by limiting ourselves to quotients G of E'_s locally free of rank n, such that the composed homomorphism $E'_{s'} \to E'_s \to G$ should be surjective (therefore bijective). But the inclusion $\mathrm{Grass}_n(S) \to \mathrm{Grass}_n(E)$ is representable by open immersion and on the other hand $\mathrm{Grass}_n(E)$ is representable by open immersion and on the other hand $\mathrm{Grass}_n(S)$ is canonically isomoprhic to the fiber bundle $V(\mathrm{Hom}_{\vartheta s}(E', E''))$.

As a result, for example, of this particular structure we may mention that if $s \in S$, then (E being locally free of finite rank) every point of $\mathrm{Grass}_n(E)$ with value sin $k(s)$ lifts to a section over a neighborhood of s. On the other hand, if $S = \mathrm{Spec}(k)$, k an infinite field, then every open non-empty subset of $\mathrm{Grass}_n(E)$ contains a k-rational point. A point of $\mathrm{Grass}_n(E)$ with values in S, i.e. a locally free quotient module G of rank n of E canonically defines a subscheme of $P(E)$, (i.e., to say) $P(G)$. Such a subscheme (but without *imposing* or specifying (Tr) the rank of G) is called a *linear subvariety* of $P(E)$ (relative to S if there is a possibility of confusion). It is therefore a projective fibration of relative dimensioni $(n-1)$ if $n \geq 1$, (and empty is $n = 0$). We immediately verify that the section of $\mathrm{Grass}_n(E)$, i.e. G is known if we know the linear subvariety corresponding to $P(E)$. In this manner the grassmanian can be interpreted as representing the functor ("linear subvarieties of relative dimension $n-1$ of P'_s") for S' variable in $n \geq 1$. It is furthermore possible to give an intrinsic characterization of the latter functor, i.e. of the notion of linear subvariety of relative dimension m and that are of "projective degree one" at every $s \in S$; this characterization will be given in a later chapter and we shall not need it at all here.

Let us again suppose that E is locally free of rank r, let E^v be its dual. Then by a polarity we find a canonical isomorphism $\mathrm{Grass}_n(E) \simeq \mathrm{Grass}_{r-n}(E^v)$ that assigns to a quotient G of E the quotient E^v/G^v of E^v. From the point of view of linear varieties to a linear variety L of relative dimension m of P there corresponds the linear dual variety L^0 of relative dimension $(r-1) + -1 - m$ of P^v, i.e. of relative codimension $(m+1)$ in P^v. (N.B. $(r-1)$ is here the relative common dimension of P and P^v over S), which we may visualize geometrically *as follows*. Let us first of all take $n = r-1$, we find an isomorphism $P(E^v) \simeq \mathrm{Grass}_{r-1}(E)$ that allows us to identify the points of P^v with values in S (let us say) as linear subvarieties of codim 1 of P (called *again hyperplanes* of P).

This says that L^0 consists of hyperplanes which *contain* the linear subvariety L of P (by which of course we mean that the points of L with value sin S' are the hyperplanes in P'_S that contain L'_S). This follows from the fact (that should have occurred at the same time as the fact that a linear subvariety L of P determines a locally free quotient G or Q [illegible] of E don't il provient that if G and g' are two locally free quotients of E (not necessarily of the same rank) then $P(G') \subset P(G)$ (as the linear subvarieties of $P(E)$), if G' is majorized by G (and the inclusion $P(G') \to P(G)$ is nothing else but the deduced morphism from $G \to G'$).

Here is a minimum of the [sorites Fr][59] which we must have at our disposal. The complete list cannot in any case be fixed (only when) until the sets of other Nos. of the present paragraph[60] are written up.

It seems to me convenient [commode Fr] to introduce also the functor Grass(E) (S') = set of quotient modules locally free (of rank not specified) of E'_S [illegible, ask AG] then Grass(E)? is representable by $\coprod_{n\geq 0}$Grass$_n(E)$. The linear subvarieties of $P(E)$ are indeed defined by sections of Grass(E) over S [NB: the rank, i.e. the relative dimension may vary if S is not connected. [slightly illegible confirm with AG].

§12 Generalization of the previous results to linear sections

Complements to notations. If $P = P(E)$, E any quasi-coherent module, we set also Grass$_n(P)$ = Grass$_{n+1}(E)$ so that Grass$_n(P)$ corresponds to linear subvarieties of dimension n in P; this is valid for $n \geq -1$ if we agree that dim $= -1$ means *empty*. If E is locally free it would be advisable to introduce

$$\mathrm{Grass}^n(P) = \mathrm{Grass}_{n-1}(P^v) = \mathrm{Grass}_n(E^v)$$

which corresponds to linear subvarieties of codimension n in P. If E is of rank $r+1$ [illegible, ask AG] P of relative dimension r we have a canonical isomoprhism Grass$^n(P)$ = Grass$_{r-n}(P)$. In what follows we suppose E fixed locally free of rank r and we are interested in linear subvarieties of P of given dimension m, thus in $\mathrm{Gr}^m = \mathrm{Gr}^m(P) = \mathrm{Gr}_m(E^v)$.

Over that prescheme we have therefore a canonical quotient G locally free of rank m of E_{Gr}, let us call it F. The natural incidence prescheme over $P_X X \mathrm{Gr}^m$, which represents the subfunctor of the product functor corresponds to the couples consisting of a section of P_S and a linear subvariety of codimension m of P'_S containing the letter, it can be made explicit therefore in the following way: let $T = P_S X \mathrm{Gr}^m$ (or if we prefer any prescheme relative or over this product), then over T we have E_T the quotient $O_T(1)$ and the sub-module locally direct factor of G^v_T. We consider the composition of the canonical homomorphisms $G^v_T \to O_T(1)$ which by transposition corresponds also to the compared analogous homomorphism of the sub-module $O_T(-1)$ of E^v_T into the quotient G^v_T: $O_T(-1) \to G_T$ and may also be

[59] Ask A.G. or Deligne about the best word
[60] What is A.G.'s meaning of paragraph?(Tr)

considered as defined by a section of $G_T(1)$, $\phi^m \in \Gamma(T, G_T(1))$. The incidence prescheme (resp. its inverse image in T) is nothing else but the prescheme of zeros of the one or the other homomorphism or of the section ϕ^m. We could denote the incidence prescheme by $H^{(m)}$ for $m = 1$; we recover the one from No. 1. If X is over P we may set $Y^{(m)} = X x_P H^{(m)}$ and define by this the notation Y^m if ξ is a point of Gr^m with values in an S'. Therefore the Y^m_ξ are "*linear sections*" of X over P (or rather of X'_S over P'_S by linear subvarieties of codimension m of P or rather of P'_S.)

I use [or profit from] this opportunity for a notational self-criticism which could come in No. 1. This point corresponds arbitrarily to indicate an object Y that corresponds to X the letter Y to X (so that if X becomes Z we no longer understand very well what to take.) This inconvenience has already led me into some incoherent notations.

Perhaps (or maybe) in the more general context with an integer m as here suggests a reasonable solution: to write $X^{(m)}$ in place of $Y^{(m)}$, thus $X^{(1)}$ in place of Y in No. 1. In such a way we might approximately have $X^{(m)(m1)} = X^{m+m1}$. I am going to try such notation in what follows. Evidently even the exponent is open to criticism since it is current practice in algebraic geometry to denote by an exponent the dimension of the varieties which enter into play. But since we shall never make use of this type of convention, I think that we have a free hand as far as that matter is concerned.

We see immediately that in the preceding construction of $X^{(m)}$ that we have a canonical isomorphism $X^{(m)} = \mathrm{Grass}_m(F^v)$ where $F_X \cong F^*(\Omega^1_{P/S})(1)$ is the kernel of $E_X \to O_X(1)$, in particular $X^{(m)}$ is smooth over X with geometrically integral fibers. (In fact, rational varieties of dimension $(m(r-m))$.) Of course, the verification reduces to the case $X = P$ and because of this it belongs just as the previous considerations to the generalities about grassmanians (which I am sure you are going to "magnify" in a separate paragraph).

We now have a perfect analogy of the diagram from No. 1. Again a forgotten point: as a prescheme over Gr^m, $H^{(m)}$ is canonically isomorphic to $P(E_{\mathrm{Gr}}/G^v)$; it is therefore an excellent projective fibration (but of course we may not conclude this in general for $X^{(m)}$ over Gr^m).

The Proposition 2.1 [se transpose Fr] translates (?) without change. In 2.2 it should read: it is necessary and sufficient that for every $x \in Z$ we have $\dim x \leq m-1$. For the proof we may, for example, restrict ourselves to 2.6 by considering a generic linear variety of codimension m as the intersection of m independent generic hyperplanes. Dieudonne demerdetur [Latin] – (or is it slightly off color French [Tr])

From the writing up point of view if (as seems preferable to me) we make from the start m general it seems preferable to prove 2.6 at the same time, where, of course, $\dim X - 1$

is replaced by dim $X - m$ (and by implying that the dimension < 0 m the formula means that the considered set is empty).

Corollary 2.3 is read by replacing 'finite' by "of dimension $\leq m-1$." Corollary 2.4 [similar]. The same for 2.5, replacing dim $f(X_i) > 0$ by dim $f(X_i) \geq m$ and the same change in 2.7.

the Proposition 2.8 remains true as stated in 2.9 replace finite by dim $\leq m-1$. The same for 2.10, 2.11.

The statement 2.12 remains valid as such with a proof essentially unchanged (compare also further down comments to No. 8); 2.13 replace finite by dim $\leq m-1$. The 2.14 stays valid as such, 2.15 by replacing finite by dim $\leq m-1$. 2.16 is valid as stated in 2.17 replace finite by dim $\leq m-1$. 2.18 as it is.

For 3.1, we can state it for any m, supposing tht dim $f(X) \geq m+1$, but I propose to keep the principal statement in the case of a hypersurface and to give the general case as a corollary as a remark (it can be deduced immediately by the usual procedure of taking independent generic hyperplanes).

At least it would be amusing to make explicit (state) the generalized version of Lemma 3.1.1.... For (3.2) read dim $f(X) \geq m+1$ in 3.3 replace dim $f(X_i) \geq 2$ by dim $f(X_i) \geq m+1$ and in the definition of G dim $f(Z) = 0$ by dim $f(Z) \leq m-1$.

The general considerations of No. 4 apply as such to the case of any m. The same is true about 4.2 and 4.3 by replacing in b) (v) and (vi) the dimension condition by dim $f(X) \geq m-1$ or $m+1$ (illegible, ask AG). Analogous change in 4.4 b).

[Le laius 5.1 Fr or Latin 67] goes as such. In 5.2 it is necessary to remember that ϕ becomes a section $\phi^{(m)}$ of $G_T(1)$ (where $T = X_S x \mathrm{Gr}^m$) inducing the sections $\phi^{(m)}\xi$ of $O_{X_{k(\xi)}}(1)_{k(\xi)} \otimes G(\xi)$ (for $\xi \in \mathrm{Gr}^m$).

But in general we shall explain in par. 19 that if we have a section ϕ of a locally free module of rank n over a prescheme this means that such a section if F regular for a given module F in terms of a local basis, this means that we have an F-regular sequence of m sections of O_X (and it will be necessary to verify that this is independent of the chosen basis). In the case $m = 1$ we have the intrinsic evident interpretation mentioned in 5.2. With this language convention 5.3 remains valid as such, also 5.4 the same.

The first part of Remark 5.5 admits a generalization to the case of m arbitrary: If $F_S$is (S_m) then the condition of regularity mentioned for ϕ_ξ can be expressed in a purely dimensional manner.

The second part of Remark 5.5 is valid as such for any m. Theorem 5.6 extends as such, so does 5.7.

Proposition 6.1. *Read* $\dim f(X_i) \geq m + 1$ *and later* $\dim f(Z) \leq m - 1$.

The laius (speech) [Fr or Latin] (speech) [. 69] general of 7.1 are valid as such in the case of any m. 7.2, 7.3 mutatis mutandis [Latin] (pay attention in 7.2 to the notation m, confusing there), on the other hand in the proof of 7.4 we no longer need to proceed closer and closer but we may take straightaway a linear section of codimension $m = n$.

In 8.2 replace condition $\dim Y_\xi > \dim X$ by $\dim Y_\xi > \dim X - m$ [illegible, ask AG] and the hypothesis $\dim f(X_i) > 0$ by $\dim f(X_i) \geq m$. The analogous modification is in the sequel to 8.2. Since 8.3 gives an example there is no point in changing it so we keep $m = 1$.

I leave it as an exercise to you [Dieudonné or Blass] [Tr] to find good statements for any m corresponding to 8.4, 8.5, and 8.6 (pages 30, 31, 32). It is not necessary to do this exercise unless you feel like doing it.

I think that essentially all the developments of No. 8 except 8.6 can be adapted to the case of linear sections with any m. To do it enforme [Fr] would be without a doubt quite a long and fastidious exercise. I have to admit that i do not know any applications depending in an essential manner on the analysis of this more general situation so we are not really obligated to include these developments in these Elements. On the other hand, experience proves that the fact of writing up in this more general context obligates often to better 'devisser' (unscrew) et fait mieux comprehendre le fourbis (the whole caboose) and often sans beaucoup plus de mal, in addition a certain number of syntax exercises in a property geometric context like here will do no harm and of course it is not at all excluded that we will one day use it or need it and we will be happy to find it. Still I leave up to you the whole decision about this subject and I restrict myself simply to summarize simply the statements that we could perhaps give in this connection (a ce proper).

Let us again assume that $f\colon X \to P$ is unramified and that X is smooth over S with components of dimension $\geq m$. Then $X^{(m)} = V^0$ we distinguish therefore the sub-prescheme $X^{(m)\mathrm{sing}} = V^1$ of the singular zeros of ϕ^m relative to Grass^m, which is also formed geometrically from pairs (x, ξ) such that the linear variety L_ξ cuts excessively the tangent space to X at x (considered as linear subvarieties of P), i.e. such that the two spaces do not generate all of P. Contrary to what happens for $m = 1$, if m is arbitrary the morphism $X^{(m)\mathrm{sing}} \to X$ is not in general smooth since the variety of L which pass through x [illegible, ask AG] and cut excessively a given linear subvariety $T \ni x$ is not in general smooth over k: this variety [illegible, ask AG] only the loneve of the subvariety smooth formed by ξ such that the dimension of $T \cap L_\xi$ is just one more than the "normal" dimensioin $(n - m)$ ($n = \dim T$, $m = \operatorname{codim} L$). V [page 71 Fr]. Barring an error, the set (contained in the relative supersingular set) V'' introduced in par. 16 (complements) is nothing else but the set formed by the couples (x, L_ξ) such that the dimension of $T_x \cap L_\xi$ is $\geq n-m+2$ so that $V'-V''$ is smooth over S and barring an error it is exactly the same as the set of smooth points of V' over X. (The verification of this point requires a study of the filtration of the Grassman scheme according to the dimensions of intersection with L variable and T fixed, barring an error we find that the following notch of [71] the filtration is formed exactly of the non-smooth points of the previous notch (*) [Fr] (stratum???)[61], when we define the filtration not just set theoretically but also scheme-theoretically using the lemma from page 16 of the complements to par. 16.[62] This study would form therefore one of the No. of a "geometric" paragraph devoted to grassmanians.)

If we also define $V^{(k)}$ as the sub-scheme of $X^{(m)}$ corresponding to dim $T_x \cap L \geq n-m+k$ we find by an immediate calculation that dim $\text{Grass}^m(P) - \dim V^{(k)} = (k-1)(n-m)+k^2$ at least for the reasonable restrictions $k \leq m$, $k \leq r-m$, up to an error of calculation. (NB this follows more generally from a calculation of the dimensions of the "cells" which intervene in the filtration of the grassmanian which was alluded to above).

For $k=2$, we find a difference of dimension ≥ 4, so that the image of V'' in Grass (illegible ask AG) is of codimension ≥ 4 so that if we are interested in what happens outside of subsets of the Grass of codimension ≥ 2 we may forget V''.

On the other hand, in$X_{\text{Grass}^m} - V''$ over Grass^m the situation is the one of the good case anticipated in the complements to par. 16. Relative to the base scheme S: $V'-V''$ is indeed smooth over S (being such over X) of relative dimension equal to one less than that of Grass^m over S (as we see by putting $k=1$ in the above formula). Thus the results of the loc cit [Latin] apply, in particular we find the fact that the set of supersingular points of ϕ^m relative to Grass^m is nothing else but $V'' \cup V^2$ where V^2 is the sub-prescheme of ramification of $V'-V'' \to \text{Grass}^m$. We may therefore say that outside of V'' the supersingular zeros result from collapsing (collapsing together) of at least two ordinary singular zeros. (but we do not have to say this).

In such a way we have essentially the equivalent of 8.7 a) and b). It should be possible to give an equivalent condition for 8.7 c) by using the explicit description of the tangent bundle to Grass^m (analogous to the caee $m=1$) it implies [illegible] that for a geometric point of $V'-V''$ unramified over Grass^m to know its image in Grass^m implies knowing its image in P in view of the fact that the first image is a smooth point of the closed image of V' in Grass^m (we assume S is the spectrum of a field). I could give a more precise statement upon request.

[61] Ask A.G.

[62] A.G.please help locate that reference

Once we grant this we have the evident corollaries generalizing 8.8, 8.9, 8.11. It is without a doubt also possible to announce in the case of m arbitrary the other propositions of paragraph 8.

If this demands additional effort of writing up we could give up this generalization, even if we include the previous differential developments.

The same is true about the results of No. 9.

As for No. 10, the situation studied there generalizes to the case of any m in the following manner. We fix a linar subvariety C of P of codimension $(m+1)$ and we consider the projective space Q of linear subvarieties $LofP$ of codimension m passing through C. Q is a close subscheme of Grass^m, in particular we could construct $X_Q^{(m)}$ which we propose to study.

A first point, which has to be in any case to figure in the text is that $X_Q^{(m)} \to X$ is again birational at least if C cuts "regularly" X and precisely $X_Q^{(m)}$ is in this case canonically isomoprhic to the prescheme deduced from X by blowing up Xx_PC: the proof of this fact is nothing else but 10.2, via 10.3. A second point which is of some interest but which we do not absolutely have to include consists in saying that if we choose C "sufficiently general", then $X_Q^{(m)} \to Q$ has certain pleasant properties, the most classical one being this: X being assumed smooth over $S = \mathrm{Spec}$ and of dim $n \geq m$ and proper and geometrically irreducible. Then, for 'sufficiently general' C the set T of $\xi \in Q$ for which $X^{(m)}$ is not smooth of dimension $(n-m)$ over $K(\xi)$ is geometrically irreducible over $K(\xi)$ and of codimensioin one in Q and the set T' of $\xi \in T$ for which $X^{(m)}$ is "supersingular" at least one point is rare (nowhere dense) in T; finally, if $F\colon X \to P$ is an immersion, then after extending T' a little, for every $\xi \in T - T'$ there is exactly one non-smooth point in $X_\xi^{(m)}$ and the latter is rational over $k(\xi)$. I forgot to specify in the statement that we assume $X \to P$ unramified and that we have to initially replace f by $\phi_n f$, $n \geq 2$ (where ϕ_n is defined in 9.1). The most natural way of proving this statement seems to be to use the subscheme Z (denoted T in 8.8) of Grass^m such that X^m is "singular": we see that, under the given conditions, it is geometrically irreducible of codimension one and that the subscheme Z^1 corresponding to $X^{(m)}$ supersingular is nowhere dense.

It remains, therefore, to prove a lemma of the following nature: let Z be a closed subset of Grass^m of codimension q then defining $Q(C)$ in terms of C as above for every C "sufficiently general" the intersection $Q(Z) \cap Z$ is of codimension $\geq q$ in $Q(C)$ also if Z is geometrically irreducible [illegible, ask AG] [itou???] $Q(C) \cap Z$ if Z is "sufficiently general."

§13 Elementary morphisms and the Theorem of M. Artin

Definition 13.1. A morphism $f\colon X \to Y$ of prescheme is called an "elementary morphism" if X is Y-isomorphic to a prescheme of the form $X' - Z$ where $X' \to Y$ is a smooth projective morphism with geometrically connected fibers of dimension one and where Z is closed sub-prescheme of X' such that the morphism $Z \to Y$ is étale surjective and of constant degree. A morphism is called *polyelementary* if it is a composition of elementary morphisms. A prescheme X over a field k is called *polyelementary* (over k) if the structural morphism $X \to \mathrm{Spec}(k)$ is poly-elementary.[63]

Theorem 13.2 (M. Artin)**.**

Let X be a geometrically irreducible prescheme over a field k, perfect and infinite, x a smooth point of X then x admits a fundamental system of open polyelementary neighborhoods.

Replacing X by a given neighborhood of x, it is enough to prove that there exists an open elementary neighborhood of x in X (??? Is that last letter correct?; illegible).

Arguing by induction on the dimension n of X, we are reduced to proving that if $n > 0$ then there exists an open neighborhood U of x and an elementary morphism $f\colon U \to V$, V being a smooth scheme over k (necessarily geometrically irreducible and of dimension $n-1$). (The case $n = 0$ is evidently trivial given that then X is isomorphic to $\mathrm{Spec}(k)$ which is polyelementary over $\mathrm{Spec}(k)$ taking into account the fact that in 13.1 we do not exclude the compsoition of the empty family of morphisms.) We should mention it in one way or another in 13.1. The necessity to assume that k is first of all perfect appears already in the case $n = 1$ where we take for X' the projective normal canonical model of the function field K of X (cf. Chap II, Par. 7)[64] the fact that k is perfect insures that X' is smooth over k (since X' is in every case regular) and it also insures that $Z = X' - X$ with the induced reduced structure is étale over k. Let us now treat the general case so that it is permissible to assume $n \geq 2$.

We may obviously suppose that X is affine, therefore quasi-projective. Further by replacing X by a projective closure we may assume that X is projective always under the reservation to prove that every neighborhood contains an open neighborhood U that allows an elementary morphism $U \to V$. Also, replacing X by its normalization (finite over X, therefore projective, *ref*)[65] which does not change the neighborhood of x, we may assume tht X is normal; therefore, k being perfect, geometrically normal over k.

[63] Reminder to Blass: ask Artin about terminology. cf. p. 117 Milne and SGA IV [Tr]
[64] EGA II, Yes 7.4 [Tr]
[65] EGA II, I think [Tr]

The benefit of this hypothesis is that the set Z of points of X where X is not smooth over k is of codim ≥ 2. Let us choose a projective immersion $i: X \to P^r$, as we obtain a fundamental system of neighborhoods of x in P^r by taking the sections not vanishing at x of the various $O_P^{(n)}$, $n > 0$, and we conclude that every neighborhood of x contains a neighborhood of the form $X - Y$ where Y is a closed subset of X containing Z,purely of dimension $(n-1)$ and such tht $x \notin Y$.

We give Y the reduced induced structure such that (k being perfect) the singular set of Y is of dimension $\leq n-2$. By enlarging the previous set Z we find a closed subset $Z \subset Y$ of dimension $\leq n-2$ containing the geometrically singular set of X and of Y.

The idea of the proof is to fiber X by its intersections with linear subvarieties L of P of codimension $(n-1)$ containing a given linear subvariety C of codimension n. To this end we will need the following:

Lemma 13.3. *With the preceding notations for X, Y, Z, $(X \supset Y \subset Z)$ closed subschemes of P_k^r of dimension n, $n-1$ and $\leq n-2$, $X - Z$, $Y - Z$ smooth, Z of dimension $\leq n-2$ et quitte (if needed)*[66] *(if k is of characteristic $p > 0$ by replacing the projective immersion $i: X \to P^r$ by any "multiple" ϕ_n^i $(n \geq 2)$ [some?] as in No. 9, there exists a linear subvariety L_0 of P^r of codimension $(n-1)$ and having the following virtues (good properties))*

a) $L_0 \cap Z = \phi = \emptyset$

b) $L_0 \cap X$ *is smooth of dimension 1*

c) $L_0 \cap Y$ *is smooth of dimension 0.*

(N.B. k denotes an infinite field *without the necessity of being perfect here.*) Let us assume this lemma and let us show how we can deduce the existence of an open neighborhood U of x contained in $X - Y$ and allowing an elementary morphism $U \to V$.

There exists a linar subvariety C of L of codimension n in P^r, i.e. of codimension 1 in L not meting the finite set $\{L_0 \cap Y\} \cup \{x\}$. Let $T = X \cap C$ so that T is a subscheme of X étale over k, non-empty and not contianing x and *disjoint* from Y. Let us consider on the other hand the subscheme Q of $\mathrm{Grass}^{n-1}(p)$ corresponding to linear subvarieties of P^r containing C such that Q is a projective space of dimension $(n-1)$; in particular it is smooth over k and of dimension $(n-1)$. Then L_0 corresponds to a point ξ of $Q(k)$. Let us consider, on the other hand, (with the general notations introduced elsewhere) the inverse image X_Q^{n-1} of X^{n-1} by the immersion $Q \to \mathrm{Grass}^{n-1}$ and also the inverse images Y_Q^{n-1} and ${}_Q^{n-1}$ which are also closed disjoint subschemes of X_Q^{n-1}, let p, q, r be the structural projections of these schemes to Q. Then by assumption essentially p is smooth *at the points*

[66] Fr

lying over ξ_0, and q is étale at the points *lying over* ξ_0; this is also the case for r as we see that T_Q^{n-1} is nothing else but Tx_kQ (Q isomorhism). Finally the morphism p is proper, and taking into account that X is geometrically connected, the fibers of p are geometrically connected (Bertini's theorem). Consequently, there exists an open neighborhood U of Q in X such that $X_Q^{n-1} \mid V = X'$ is proper and smooth over V with geometrically connected fibers, and since the fiber of Q is nothing else but $X \cap L_0$, it is of dimension 1, we may suppose that the fibers of X' over V are all of dimension 1. Finally, taking V sufficiently small, we may suppose that $Y_Q^{n-1} \mid V$ and $T_Q^{n-1} \mid V$ are étale over V so tht the *sum* prescheme of Z' of the two (which is identified with a closed prescheme of X') is étale over V. Consequently, putting $U = X' - Z'$, the morphism $U \to V$ is an elementary morphism. But U is also an open subset of $X'' = X_Q^{n-1} - Y_Q^{n-1} - Z_Q^{n-1}$, the inverse image of $X - Y - T$ in X_Q^{n-1}; on the other hand, $X'' \to X - Y - T$ is an isomorphism (*since* (*car*) $X_Q^{n-1} - T_Q^{n-1} \to X - T$ is an isomorphism). Therefore U is identified to an open subset of $X - Y - T$, an open subset containing, furthermore, $L_0 \cap X$ and a fortiori x. This is the desired neighborhood of x contained in $X - Y$.

It remains only to prove Lemma 13.3. As usual, it suffices to prove that the generic linear subvariety of codimension $(n-1)$ passing through x called L has properties a), b), c). To prove a) as well as the dimensional content of b) and c), this follows immediately from 2.3 (reviewed and corrected in No. 12) applied (as in a reasoning already done in No. 8) to projective space of straight lines passing through X and the image of Z [*dans ledit*] by a conic projection from x. [It might be useful in addition to make explicit certain results obtained by this method concerning the linear sections by linear subvarieties subject to the condition of passing through a fixed linear subvariety. In the text or in a separate No.] For the smoothness in b) and c) we can because of a) replace X and Y respectively by $X - Z$ and $Y - Z$ which are smooth and we are reduced to proving this: Let $f\colon X \to P$ be an unramified morphism such that x does not belong to the image of any component of X of dimension $< m$ (*irreducible component*?) with X smooth over k and let $X \in P(k)$ then if η is the generic point of the subgrassmanian of $\mathrm{Grass}^m P$ formed from linear varieties L of codimension m passing through x, $X^{(m)}$ is smooth over k at least if k is of characteristic zero and the opposite case, on condition by replacing f by $Q_n f$, n an integer ≥ 2.

This is a regret to No. 9, which itself follows from the regrets following No. 8: With the notations of 8.8 (supposing that X is irreducible, which is legal for the p^0-problem (?) that we are discussing) if we have $\mathrm{codim} T \geq 2$ or if $h^{\mathrm{sing}} \to T$ is generically étale (condition that is automatically satisfied if k is of characteristic zero or on condition of replacing f by $\Phi_n f$ with $n \geq 2$, cf No. 9, then for the hyperplane H_η passing through a generic x $X_\eta^{(1)}$ is smooth of dimension $(n-1)$, except in the case where we have $f(x) = \{x\}$ (thus $n = 0$).

This result allowed [admis Fr] which liquidates evidently the special case $m = 1$ of our regrets, we obtain immediately the case of a general m by induction on m by noticing that up to a change of basis $X_\eta^{(m)}$ is obtained by taking an L' of codimension $(m-1)$ passing through x, L' and H being generic independent for these properties (i.e. in orthodox terms we place ourselves at the generic point of the scheme of pairs (L', H) and by taking the linear sections by H which is smooth by inductive assumption.) This type of reasoning already used to generalize 2.6, for example to linear sections of any codimension m deserves to be made explicit one good time in general so that we may refer to it without entering every time into the details, a bit heavy [Fr] [ ] of a presentation (enforme).

It remains to prove the corollary announced of 8.8 in the case $m = 1$. If $\operatorname{codim} T \geq 2$, since on the other hand the hyperplane Q of p^v of H such that $x \in H_\xi$ is of codimension 1, its generic point η cannot be an element of T and on gagne [Fr]. (we are done?) In the case $\operatorname{codim} T = 1$, since T is irreducible we cannot have $\eta \in T$ under $Q = T$, i.e. in geometric terms (supposing k algebraically closed which is loegal for every $z \in X$ the tangent space to X at z (or also plutot [Fr] is image by f'_z goes through x. Let us prove that this cannot happen unless $X^{\text{sing}} \to T$ is generically étale, i.e. unless we are under the condition so f8.8, except in the case $f(x) = \{x\}$ thus X of diemnsion zero. Indeed 8.7 c) (which expresses essentially the symmetry in the relation between X and its "dual" T) implies therefore that for almost every point $z \in X(k)$ $f(z)$ is orthogonal to the tangent space to T at a certain point this (since $T = Q$) orthogonal to Q, where $f(z) = x$ hence $f(x) = \{x\}$. this proves our regret, thus 13.2.

N.B. The reasoning does not go through if we replace x by a linear sub-variety X of $\dim > 0$, and if we subject H to passing through C; indeed, there is no reason to suppose (taking for example dim $(= r - 2$ the greatest possible for which H can still effectively vary) without supposing (?) that T contains the straight linear C^0 taking for example a non-singular quadric in p in any characteristic. But it is possible that such phenomena cannot happen anymore for $\phi_n f$, $n \geq 2$; we could pose the questions as a remark in No. 9.

Remark 13.4.

a) We have already observed that in the hypothesis that k should be perfect is essential for the validity of 13.2. On the contrary, it is plausible that the hypothesis k is infinite is superfluous too strong. We shall not try an ad hoc reasoning for the case where k is finite and we only note that in this case the application of 13.2 to the algebraic closure of k and usual arguments show tht we may find a finite extension k' of k such that for the point of S'_k over x there exist open polyelementary neighborhoods realtive to k'.

b) If in 13.2 we abandon the hypothesis that X is geometrically irreducible the conclusion obviously does not remain the same (since an algebraic poly-elementary scheme is geometrically irreducible). It holds, however, in a weaker form which is obtained by omitting in definition 13.1 the word “connected” this is shown by the proof that we have given.

c) (To be possibly included in the statement of 13.2) Let with the notations of 13.2 Φ be a finite subset of X formed by smooth points of X and let us suppose that Φ is contained in an open affine of X. Then Φ allows a fundamental system of open polyelementary neighborhoods. Evidently we may suppose that Φ consists of closed points. The proof is essentially the same except that in formula 13.3 in slightly different form we have: there exists a linear subvariety C of P^r of codimension not meeting $\Phi \cup Y$ and such that for every $x_i \in \Phi(k)$, the linear subvariety of L_i of codimension $(n-1)$ generated by C at x_i has the properties a), b), c) of 13.3. To verify this point we note that it suffices to verify that the generic C has the above-mentioned properties since for such a ??????? each L_i is generic among the L of codimension n passing through x_i so that we can apply 13.3 in the initial form (or at least in the form that we have proven which was [Fr]: every L_0 sufficiently general passing through x has properties a), b), c).

d) By proceeding as explained in 7.1, we may give variants of 13.2 in the case where we replace the base field k by a general base Y prescheme. Let us remark the following (without proof): Let $f\colon X \to Y$ be a flat projective morphism with geometrically irreducible and (R_2) fibers, S' a subscheme of X finite over S, $x \in S$, suppose that for every $x \in S'$ over s, X_s is smooth over $k(s)$ at x. Then there exists an open neighborhood U of s and an open neighborhood V of $S' \mid U$ in $X \mid U$ such that $V \to U$ is poly-elementary. If Y is a closed subscheme of X not meeting S' and such that the set X of points where Y is not smooth over S $\dim Z_s \leq \dim X_s - 2$ then we may above tale V containe din $X - Y$.

e) One of the reasons why 13.2 is interesting is the topological structure particularly simple of the elementary algebraic schemes U. For example if the base field is the field of complex numbers and if Y^{en} denotes the analytic space associated to U then the homotopy groups $\pi_i(U^{\mathrm{en}})$ are zero for $i \neq 1$ and π_1 is a successive extension of free groups. Thus U^{en} is a “space $K(\pi_1, 1)$” classifying for π_1, more precisely its universal covering space is homeomorphic to C_n and a fortiori is contractible and this covering is a “universal principal fibration” with group π_1.

§14 Conic Projections

N.B. We have already used Conic Projections in different contexts, notably at the end of No. 8, formulation of 10.4 and others and the "sorite" that follows should without a doubt come sooner in the beginning of the paragraph and eventually in the auxillary paragraph "grassmanian". Let $C = P(F)$ be a linear subvariety of $P(E) = P$ of relative dimension $r - m - 1$ over S, i.e. of codimension $(m + 1)$ in P so that F is a quotient of E locally free of rank $r - m$, $F = E/G$ where G is locally free of rank $m + 1$. We have defined in the algebraic way of Chapter II a morphism

$$p_c : P - C = P(E) - P(E/G) \to P(G)$$

which we will interpret geometrically and which will be called (because of the description that follows) the *conic projection* with center C. (N.B. We assume $r - m - 1$ is contained between -1 and $r - 1$, i.e. m is between 0 and r, nothing more. For this let us begin by interpreting $P(G)$ as a closed subscheme of $\mathrm{Grass}^m(P) = \mathrm{Grass}_{r-m+1}(P)$ due to the obvious homomorphism of functors $P(G) \to \mathrm{Grass}_{r-m+1}(E)$ obtained by considering for every invertible quotient G/G' of G the locally free module of rank $(r - m) + 1\, E/G'$ of E and the same after every base change). The above homomorphism of functors is a homomorphism and since the first one is proper over S the second one separated it is a closed immersion. More generally we may need to make explicit the closed immersion of grassmanians of G, i.e. of $P(G)$ (in the sense of functors) into those of E, i.e. of $P(E)$. The image (in the sense of functors) of the obtained morphism is formed from linar subvarieties L of the desired dimension of P that *contain* C. Let us denote by $Q(C)$ this image in the case that we are studying (i.e. for the dimensions specified above) and identifying $P(G)$ with $Q(C)$ the morphism of conic projection

$$p_c : P - C \to Q(C) \subset \mathrm{Grass}^m(P)$$

is nothing else but the one that associated with every sectin of $P - C$ the unique linar subvariety L of P of codimension m containing at the same time C and the given section (note, obviously that by containing a section we mean that the section factors by L). If not, we have an $f : X \to P$ it makes sense to consider the composition

$$X - f^{-1}(C) \to P - C \to Q(C)$$

which we may call conic projection of X relative to f and with center C denoted p_c^X or simply p_c. We point out that it is not in general defined over all of X, precisely it is such if and only if $f^{-1}(C) = \phi$, i.e. $f(x)$ does not meet the center of the projection C. We shall give another interpretation of this morphism in terms of construction used in previous Nos. For this with the notations introduced elsewhere let us consider

$$\begin{array}{ccc} X & \xleftarrow{q} & X^{(m)}_{Q(C)} = X^{(m)} \underset{Grass^m}{\times} Q(C) \\ & & \downarrow p \\ & & Q(C) \end{array}$$

Let us note on the other hand that q induces an isomorphism

$$q' : q^{-1}(X - f^{-1}(C)) \xrightarrow{\sim} X - f^{-1}(C)$$

and it is immediate that p_c is nothing else but $p'q'^{-1}$ where p' is the restriction of p to $q^{-1}(X - f^{-1}(C))$. We may therefore say using q' to identify purely and simply that p_c is the restriction of the morhism p to $X - f^{-1}(C) \subset X(M)^{p_c}_{Q(C)}$. For that reason it is convenient to denote again by p_c^X or p_c and to call the above morphism [(*Mettons*)] the extended conic projection of X relative to $f\colon X \to P$ with center C. In this way the properties of the restricted conic projection are reduced to those of the extended conic projection which has been systematically studied elsewhere or is supposed to have been studied[67] and it makes sense (cf. No. 10 and No. 12). The main question that arises is if $S = \mathrm{Spec}(k)$ what are the properites of the conic projection of X if we take C to be generic in $\mathrm{Grass}^{m+1}(p)$ [illegible, ask AG] which requires that we make a base change $k \to K(\eta)$, i.e. C is then indeed a linear subvariety of $X_{k(\eta)}$ from standard arguments that have already been repeated several times allow us to conclude the analogous properties for the conic projections corresponding to the points of $\mathrm{Grass}^{m+1}(P)$ belonging to an open non-empty set of the said grassmanian and finally since k is infinite (if) we conclude the existence of a (in fact of an infinity of) C defined over k, i.e. a linear subvariety of P iself (without changing the base field) giving rise to a conic projection having the properties in question. It is (will be) proper to group this type of general explainations with those of the same type given in No. 4, 7 and which we have already used more or less implicitly, for example in No. 13. It is also proper *by the way in this connection* to examine the relative properties of a sheaf F over X and taking its inverse image $F^{(m)}_{Q(C)}$ over $X^{(m)}_{Q(C)}$. It is necessary in addition in the precise situation described here to simplify the notation I propose $X(C)$ and $F(S)$ or simply $\widetilde{X}$ and $\widetilde{F}$ if there is no possibility of confusion (attention: the F is not the same as in the beginning of this No.). Grosso modo (roughly speaking) and if we, say, assume that f is an immersion the properties of the generic conic projection

[67] Ask A.G.

are very different according as to whether we assume dim $X \geq m$ or dim $X \leq m$ see dim $< m$. In what follows we consider the $C_\eta \subset P_{k(\eta)}$ corresponding to the generic point η of Grass^{m+1} and we dispense with making the interpretation of the obtained results in terms of "almost all the points ... "

To start with, we already have noticed in 5.3 (a 'catching up' *due* to the general case in No. 12) that C cuts X regularly, more precisely and more generally for every quasi-coherent F over X the section $\phi(m+1)$ of the locally free module of rank $m+1$ over $X_{k(\eta)}$ whose scheme of zeros η is C, is F-regular. By 10.2 this implies for example that the morphism $\widetilde{X}_{(C\eta)} \to X_{k(\eta)}$ identifies $X_{(C\eta)}$ with the prescheme deduced from $X_{k(\eta)}$ by blowing up the $f^{-1}(C_\eta) = X_p x C_\eta$ in the case where dim $f(X) \leq m$ we will also have $f^{-1}(C) = \phi$ and consequently $X(C_\eta) \xrightarrow{\sim} X_{k(\eta)}$ is an isomorphism (and indeed the restricted conic projection is therefore defined over all of X a priori). The question arises consequently of the dimension of the fibers of $p_c \colon \widetilde{X}(C_\eta) \to Q(C_\eta)$, and we find the flatness of this morphism. We find:

Proposition 14.l. *Let us suppose that X is irreducible, more generally that for every irreducible component X_i of X the fiber of X_i at the point $f(x_i)$ (x_i = generic point of X_i) has a dimension (independent of i), which is for example the case with $d = 0$ if $f\colon X \to P$ is quasi-finite. Then*

a) *If* dim $f(X) > m$ *then the dimension of the fibers of $p_c \colon X(C_\eta) \to Q(C_\eta)$ are all equal to* dim $X - m$.
b) *If* dim $x \leq m$ *and if the non-empty fibers of X^i over P are ??? of* dim d *then the fibers of p_c are all of dimension d ??? so p_c is finite resp. quasi-finite... $f\colon X \to P$ is finite.*

In the case a) we have already seen (I hope) that for *every* point ξ of $\text{Grass}^m(P)$ the dimension of $X_\xi^{(m)}$ is at least equal to dim $X - m$ it is such in particular if ξ gives a point of $Q(C_\eta)$. For the opposite direction inequality note that (we place ourselves over the field $k' = k(\xi)$) since $C_{\eta k'}$ L_ξ is a hyperplane of L if the dimension of $X^{(m)} = X_p x C_\eta$ will be (*is*) $\geq$ dim $X - m$ (since the base change $k(\eta) \to k'$ transforms the latter prescheme into $(X_p \times L_\xi \times \xi_L(c_{\eta k'}))$ *or* since we have in the contrary case dim $X^{(m+1)}$ = dim $X - m - 1$ by No. 2 (reviewed in No. 10). The case b) is treated in an analogous fashion if we have dim $X_p \times L \geq d+1$, or what is the same $f'_k(X_{k'}) \to L$ of dim ≥ 1 then we would have by the same argument as above that $X^{(m+1)} \neq \phi$ contrary to what we have remarked before 14.1.

Corollary 14.2. *Let us assume that X has dimension m and that* $: X \to P$ *is finite* *respectively quasi-finite, then the morphism $P_{C\eta}: X_{k(\eta)} \to Q(C_\eta)$ is finite surjective (resp. quasi-finite dominant).*

Indeed, this morphism is quasi-finite and since $\dim X_{k(\eta)} = \dim Q(C_\eta)$ *it is dominant if f is finite $p_{C\eta}$ is also finite, therefore proper, therefore surjective, since it is dominant.*

Corollary 14.3. *With the conditions of 14.1 a) if X is Cohen-Macauley the morphism $p_C: X(C_\eta) \to Q(C_\eta)$ is a Cohen-Macauley morphism and is a fortiori flat.*

For the proof compare the remark above on page 21 before 5. (Tr - correct this) which gives a result which is stronger (including 14.3 ???) taking into account that $\mathcal{C}_\xi^{(m)}$ for $\xi \in \varphi(\eta)$ are $F_{k(\eta)}$ regular.

This corollary must be modified but for simplicity we may assume that f is quasi-finite if F is a Cohen-Macauley module over X and if for every irreducible component Z of $\operatorname{Supp} F$ we have $\dim Z \geq m$ then $\widetilde{F}(C_\eta)$ is Cohen-Macauley and á fortiori flat relative to $\varphi(C_\eta)$.

We note that we cannot replace, to obtain the same conclusion p_C flat, the CM hypothesis on X by a simple dimension hypothesis. Let us for example assume that f is an immersion and that f is irreducible of dimension m, so that p_C is quasi-finite and since $X_{k(\eta)}$ and $\varphi(C_\eta)$ are irreducible of the same dimension and the second one is regular, p_C cannot be flat unless $X_{k(\eta)}$ is CM.

More delicate are the differential properties of the conic projection, notably for X smooth over k and $f: X \to P$ unramified studied in No. 12. Let us recall that outside of a subset Z of codim 1 of $Q(C)$ the morphism $p_{C\eta}$ over $\widetilde{X}(C_\eta)$ is smooth. And a more detailed analysis summarized in No. 12 shows or will show if we do not do it that if the dimensions of the components of X are $\geq m$ then outside of a subset $Z' \subset Z$ of $Q(C)$ of codimension ≥ 2, th fibers $p_C^{-1}(\xi) = X_\xi^{(m)}$ can only have at worst ordinary singular points in the geometric sense and indeed (if f is an immersion and X is geometrically irreducible) at most *one* such point, the latter being necessarily rational over $k(\xi)$ – these assertions being all valid at least if k is of characteristic 0 *or* with the condition of replacing f by$\phi_n f$ $(n \geq 2)$ as in No. 9.

It is also appropriate to give the differential properties of $P_{C\eta}$ in the case where $\dim X \leq m$ and consequently $P_{C\eta}$ is defined over $X_{k(\eta)}$. I restrict myself to indicating the following properties. The proof should be easy and is left to Dieudonné (or Blass). [Tr]

Proposition 14.4. *Let us suppose that $f: X \to P$ is unramified and that* $\dim X \leq m$. *Let T be a finite subscheme of X. Then*

a) If f is an immersion, the restriction of p_C to $T_{k(\eta)}$ is radical, i.e. "geometrically *injective". If in addition Y a closed subset of X of dimension $\leq (m-1)$ we have*

$$p_{C\eta}^{-1}(p_{C\eta}(Y_{k(\eta)})) \cap T_{k(\eta)} = \phi = \text{ empty set}$$

b) *If X is smooth at the points of T then $p_{C\eta}$ is unramified at all the points of $T_{k(\eta)}$ [illegible, ask A.G.]*

$$p_{C\eta}^{-1} p_{C\eta}(T_{k(\eta)})$$

Proposition 14.5. *Let us suppose that* $\dim X \leq m-1$, *$f\colon X \to P$ an immersion, finally X separable over k. Let Y_η be the scheme theoretic image of $X_{k(\eta)}$ in $Q(C_\eta)$. Then the induced morphism $p_{C\eta}\colon X_{k(\eta)} \to Y_\eta$ is birational and for every point x of $X_{k(\eta)}$ over a closed point of X, $p_{c\eta}$ is étale at x and in at the points of $p_{C\eta}^{-1} p_{C\eta}(x)$.*

Let us note the following consequence:

Corollary 14.6. *Let X be an algebraic projective scheme irreducible and separable of dimension n over an infinite field k. Then there exists a birational morphism of X onto a hypersurface in P^{n+1}.*

We will avoid believing, even if X is a closed smooth geometrically irreducible subset of p of dimension $m-1=n$, that the conic projection (P_c) (? Illegible) is necessarily an immersion. Indeed if k is infinite this implies that there exists a C rational over k having the same property, this that X is isomorphic to a non-singular hypersurface in P^{n+1}. But even or already for $n=1$ (thus X an algebraic projective curve smooth and connected over an algebraically closed field) it is easy to construct examples when X cannot be embedded (ne peat s'immerge) in a p^2. Also in 14.4 we will avoid confusing the given statement with the assertion (in general false) that p_c is itself a monomorphism (preceding counterexample if X is smooth if dimension m), or that p_c should be unramified. For the later point we will take to convince ourselves X a closed smooth subscheme irreducible and of dimension m (over k algebraically closed ?illegible? such that we have an $X \to Q \cong p^m$ unramified, it will be étale for reasons of dimension, but we can prove (see Ch. VIII) that this implies that $X \xrightarrow{\sim} p^m$ (p^m being simply connected). The intuitive geometric meaning of 14.4 is that the ramification set of $p_{C\eta}$ is "variable" over k more precisely the ramification set of $p_{c\xi}$ for a variable ξ in an open set of $\mathrm{Grass}^{m+1}(\bar{k})$ varies in $X(\bar{k})$ and does not admit any "fixed point"... Of course, that to justify in the present No. the passage from η generic to neighboring points of $\mathrm{Grass}^{m+1}(P)$ and also in case of need, to be able to reaccept responsibility for the general considerations of 7.1, we have to consider the diagram:

$$\begin{array}{ccc} X & \longleftarrow & \widetilde{X}(C) \\ \downarrow & & \downarrow \\ X & \longleftarrow & Q(C) \end{array}$$

obtained (with the help) using the different $C \in \mathrm{Grass}^{m+1}(S)$ and more generally the ones obtained after a base change $T \to S$ for the points $\xi \in \mathrm{Grass}^{m+1}(P)$

$$\begin{array}{ccc} X_T & \longleftarrow & \widetilde{X}(C_\xi) = X_T(C_\xi) \\ \big\downarrow & & \big\downarrow \\ T & \longleftarrow & Q(C_\xi) \end{array}$$

as deduced by base change $\xi\colon T \to \mathrm{Grass}^{m+1} J(P) = T$, of the universal diagram (relative to the canonical point of Grass^{m+1} in T):

$$\begin{array}{ccc} X_T & \longleftarrow & \widetilde{X}(C) \\ \big\downarrow & & \big\downarrow \\ T & \longleftarrow & Q(C) \end{array}$$

where C is the canonical linear subvariety of P_T. Then the above $\widetilde{X}(C_\eta) \to Q(C_\eta)$ is nothing else but the morhism of the generic fibers for the T morphism $\widetilde{X}(C) \to Q(C)$ of the latter diagram and every constructible property for the morhism of generic fibers implies the same property for neighboring fibers. From the notational point of view, Q should be considered (and even introduced) as the name of the natural morphism of functors $\mathrm{Grass}^{m+1}(P) \to$ Subschemes of $\mathrm{Grass}^{m}(P)$.

§15 Axiomatization of certain of the previous results

I think overall, tht results of No. 2 to 8 which are all mostly true under more general conditions than for the family of hyperplanes (or for hypersurfaces of given degree) in projective space. It seems to me proper to adopt an axiomatic point of view. I am not quite sure right now if we can make a generalization in this sense of Bertini-Zariski (therefore of the result sof No. 4 and 6) and I have written to the (competent people Tr) authorities on this subject (Serre-Zariski) to ask them if they had a knowledge of such an extension. I have anyway the impression in effect that the hypotheses of simple differential nature of the type of those given above should suffice to imply Bertini-Zariski. If the (experts) competent people cannot inform us in a satisfactory manner, we should try to clear the matter up by our own means. We start from a commutative diagram of morphisms of finite presentation

$$\begin{array}{ccc} P & \longleftarrow & P \\ \big\downarrow & & \big\downarrow \\ S & \longleftarrow & G \end{array}$$

(in the case of the principal application P is a projective fibration, G a deduced grassmanian (grasmanienne-adjective!)[68] P the incidence prescheme. In the most important cases the corresponding morphism $P \to P_S \times G$ should be a closed immersion and we consider G as a parameter scheme of a family of closed fiber subpreschemes of P over S, more precisely if $\xi \in G$ then P_ξ is a closed subprescheme of P_s, $k(\xi)$ where s is the point of S over ξ besides for most statements in this context we have no doubt $S = \mathrm{Spec}(k)$.) In the general case we may again consider G as a parameter scheme of a family of preschemes over the fibers of P over S with ξ corresponding to P_ξ over P_s, $k(\xi)$. Of course, in place of taking for ξ a point (absolute) of G we may also take a point with values in an S-prescheme T, and we obtain then $P_\xi \to B_T$ (T morphism which si a closed immersion in the case presented above).

If $f\colon X \to P$ is a morphism we put $X = X_P \times P$ and we obtain a diagram of the same type as the preceding square.

$$\begin{array}{ccc} X & \longleftarrow & X \\ \downarrow & & \downarrow \\ S & \longleftarrow & G \end{array}$$

It is therefore evident that all the questions studied in No. 2 to 8 preserve a meaning in the general context that we just enunciated and there is a good reason to[69] the axiomatic conditions that insure the conclusions drawn in the above Nos.

We will assume that P *and* G *are flat over* S, G being with geometrically irreducible fibers (to be able to consider the generic points!) of dimension N, the morphism $P \to P$ is assumed to be smooth with geometrically irreducible fibers of dimension $N-m$. Therefore the morphism $X \to X$ has the same properties. All the properties mentioned [(*illegible and long*)][70] are stable under base change (preserved by) over S and can in particular be applied to the fibers.

Let us assume initially $S = \mathrm{Spec}(k)$. Let Z be a closed subset of X of dim d so that its inverse image Z in X is a closed subset of dimension $d + (N - m) = N + d - m$. If $d < m$ then Z is of dimension $< N$ so that $Z \to G$ cannot be dominant therefore if η is a generic point of G we have $Z_\eta = \phi$; indeed this reasoning shows even (by replacing Z by $\overline{f(Z)}$) tht if dim $f(X) < m$ then $Z_\eta = \phi$. We want a condition on (D) insuring that if dim $f(Z) \geq m$ then $Z_\eta \neq \phi$. It seems that when it must form a primitive axiom in this situation (in the setting of No. 2.2 it would result from a global argument rather (quite) special) for every closed irreducible subset Z of P of dimension m, $Z_\eta = \phi$.

[68]possibly grassmanian fibration [Tr].
[69]Fr degager + unravel, make explicit ?? [Tr]
[70]illegible later

Let us again take a closed subset Z of X such that dim $f(Z) \geq m$ we see that $Z_\eta \to G$ is dominant and consequently Z is of dimension equal to dim $Z -$ dim $G =$ dim $Z - m$. These properties allow us to develop in the present context the resdults corresponding to 2.1 and 2.11. There is a condition over [illegible] (insuring the validity of 2.12, i.e. that if X is smooth then X is also such if we assume $f\colon X \to P$ unramified). We assume now that P is *smooth* over k, $P \to P_k \times G$ quasi-finite, and that the following condition is satisfied (where we assume k algebraically closed) for every $x \in P(k)$ and for every vector subspace V of dimension $n \geq m$ of the tangent space $T_x(P)$ to P at x, we consider the set $E(x, V)$ of $\xi \in G(k)$ such that P has a point over x *not* satisfying the following set of conditions: P_ξ is smooth at z, the tangent morphism of $P_\xi \to P$ at z mapping $T_z(P) \to T_x(P)$ is injective (i.e. $P_\xi \to P$ unramified at z) and its image is "transversal" to V, i.e. its sum with V is $T_x(P)$. Then $E(x, V)$ (which weknow to be the trace of a constructible well defined set of G in $G(k)$???) is of dimension $\leq N - n - 1$. using [(*Moyennant*)] this condition, an application of the Jacobian criterion and a dimension count shows that the closed subset E of points x of X such tht $X \to G$ should be non-smooth at x or $P \to G$is not smooth at $f(x)$ or $P \to P$ is ramified at $f(x)$ is of dimension $\leq n + (N - n - 1) = N - 1$ (X being smooth everywhere of dimension n). Therefore dim $E < N =$ dim so that $E_\eta = \phi$ and a fortiori X_η is smooth over $k(\eta)$ and the developments of No. 5 are evidently valid in this current context.

The passage in No. 4 from a generic section to a general section and the developments of No. 5 are evidently valid in the present context (but are at this point tautologies or a reformulation of paraagraphs 8, 9, 12 which we hesitate to announce in their form). Also the development sof 7.1 valid in every case if k is algebraically closed (and even if k is simply infinite if we assume G rational over k) and the special cases 7.2, 7.3' quant (???) the result 7.4 is evidently an application of special nature for the situation of hyperplane sections. As I have said, the numbers 3 and 6 are (suspended pending) to the extension of the theorem of Zariski.

It remains to extend also the results of No. 8 (reconsidered in No. 12) which take on such a more pleasant allure. I advise you to begin formulating these results in this context in rying to go as far as possible in this way. I have the impression that we have to be able to recover at least that is not a direct consequence of 8.7 c) (even we could attempt to abstract the axiomatic conditions that allow you to go through a variant of 8.7. I limit myself to these recommendations but I am ready to go back to these with more details if you have special difficulties.

Part II [Tr]

§16 New EGA V

New # Singular and supersingular set of a function and differential criteria

This No. will be used in Par. 20 of hyperplane sections but its natural place is I think [Grothendieck] in Par. 16.

Definition 1. Let X be a regular prescheme ϕ a section of O_X. A point $x \in X$ is called a *singular* zero (or root) of ϕ if we have $\phi_x \in m_X^2$, it is called a *supersingular* zero if it is a singular zero (or root) and if in addition the element of $m_X^2/m_X^3 \cong \mathrm{Sym}(m_X/m_X^2)$ which it defines interpreted as a quadratic form on the dual t_x of m_X/m_X^2 over $k(x)$ is a degenerate form. (A singular zero (or root) which is not supersingular is sometimes called an *ordinary singular zero.*)

Remarks 2. If $x \in V(\phi)$ then x is a non-singular zero of ϕ if and only if $\phi_x \neq 0$ and x is a non-singular point, i.e. x is a regular point of $V(\phi)$, i.e. if and only if x is a regular point of $V(\phi)$ and $V(\phi) \neq X$ in a neighborhood of x.

Definition 3. Let X be a smooth prescheme over a field k, ϕ a section of O_X, $x \in V(\phi)$. We say that x is a *geometrically singular* (resp. *geometrically supersingular*) zero of ϕ relative to k if for every extension k' of k and every point ξ of X with values in k, localized at x, the corresponding point x' of X'_k is a singular zero (resp. supersingular) of ϕ'_k.

Remark 4.

a) From the criterion that will be developed later it follows that in Definition 3 it suffices to test with a single point with values in a k' we *can* for example take $k' = k(x)$ or $\overline{k(x)}$ and the canonical point with values in such k'.

b) It follows from Remark 2 that x is geometrically non-singular for ϕ if and onlyif $\phi_x \neq 0$ and $V(\phi)$ is *smooth* at x.

c) Let us suppose that we have a prescheme X smooth over another one Y a section ϕ of O_X and an $x \in V(\phi)$ then we say that x is a singular zero (or supersingular) relative to Y if it is such relative to $k(s)$ over the fiber X_s (s being the image of x in Y).

d) Under the conditions of Definition 1, we see at once that the singularity resp. supersingularity of an $x \in V(\phi)$ for ϕ is not modified if we replace ϕ by $\phi' = u\phi$ where u is a unit at x. It follows, furthermore, that Definition 1 and consequently also Definition 3 can be extended in an evident way to the case where ϕ is a section of an invertible module L (in a way so that to reproduce the initial definition for $L = O_X$). Let X be a prescheme smooth over another one Y and let ϕ be a section of O_X whence a section

$d^2_{X/Y}\phi$ of $P^2_{X/Y}$, which reduces to a section of $d^1_{X/Y}\phi$ of $P^1_{X/Y}$ which itself reduces to a section $d^0_\phi = \phi$ of $P_{X/Y} = O_X$. This assumed, we have:

Proposition 5. *The set of zeros of d^0_ϕ (resp. d^1_ϕ) is respectively equal to the set $V(\phi)$ of zeros of ϕ (resp. to the set $V(\phi)^{\text{sing}}$ of singular zeros of ϕ relative to S). The first assertion is trivial. The second one is nothing else but the Jacobian criterion or if one prefers it results from the canonical isomorhism $m_X/m_X^2 \cong \Omega^1_{X/k(x)}$ since x is a point rational over k of a prescheme x over k.*

Let us note that $\mathrm{gr}^1(P^1_{X/Y}) \cong \Omega^1_{X/Y}$ so that consequently the restriction $d^1\phi \mid V(\phi)$ can be interpreted as a section of $\Omega^1_{X/Y} \otimes O_{V(\phi)}$ which is nothing else than the restriction of $d_{X/Y}\phi$ to $V(\phi)$. We can therefore consider the prescheme of zeros of this section which we denote $V(\phi)^{\text{sing}}$, and whose underlying set is nothing else but the set of zeros of ϕ singular relative to Y by Proposition 5. (N.B. If Ψ is a section of a locally feee module E of finite type over a prescheme X, it defines in an obvious way; the prescheme of zeros of Ψ for example as defined by the image ideal of $E^V \to O_X$ transpose of Ψ; if $E = O_X^n$ and $\Psi = (\Psi_1, \dots, \Psi_n)$ then this ideal is nothing else but $\Sigma\Psi_i O_X$ which defines $V(\Psi_1, \dots, \Psi_n)$. Now taking the restriction $d^2\phi \mid V(\phi)^{\text{sing}}$ and noting that $\mathrm{gr}^2(P^2_{X/Y}) \cong \mathrm{Sym}^2(\Omega^1_{X/Y})$, we find a canonical section $M(\phi)$ of $\mathrm{Sym}^2(\Omega^1_{X/Y}) \otimes O_V^{\text{sing}}$. We verify immeditely (taking a point of X with value in a field...) that this section is precisely the one which determines the quadratic form given in Definition 1 (in the case of a X_k deduced from X/S by $\mathrm{Spec}(k) \to S$). We deduce a description of the set $V(\phi)^{\text{sup sing}}$ in terms of this section in the following manner: interpreting $M(\phi)$ as defining a homomorphism

$$M(\phi)' \colon G_{X/Y} \otimes_{O_X} O_V^{\text{sing}} \to \Omega^1_{X/Y} \otimes_{O_X} O_V^{\text{sing}}$$

we must take the set of points where this homomorphism is not an isomorphism. This proves in particular that $V(\phi)^{\text{sup sing}}$ is a closed set. We can make the latter precise by introducing

$$D(\phi) = \det\, M(\phi) \in \Gamma(\Omega^d_{X/Y})^{\otimes 2} \otimes O_V^{\text{sing}}$$

and supposing that X has relative dimension d over Y at every point. We need to denote by $V(\phi)^{\text{sup sing}}$ the closed subschemes of $V(\phi)^{\text{sing}}$ therefore of X defined by the vanishing of this section (of an invertible module here) thus the underlying set is that which is needed. It is proper to summarize this construction in a Proposition 6 (to be supplied by the editor [Tr]).

In the general case we can say nothing more precise about $V(\phi)^{\text{sing}}$ and $V(\phi)^{\text{sup sing}}$. We now examine the particular case interesting for some applications. We suppose that Y

is also smooth over a prescheme S with constant relative dimension m to fix ideas. Also, we suppose that $V(\phi)^{\text{sing}}$ which we denote simply V' for simplicity defined by the vanishing of the section d^1 of the module $P^1_{X/Y}$ locally free of rank $d+1$ is smooth over S of relative dimension $(m+1)-(d+1)=m-1$ (N.B. Note of course that the notations $V(\phi)^{\text{sing}}$ and $V(\phi)^{\text{sup sing}}$ are ambiguous in the sense that there does not intervent the prescheme to which they are related; in the actual case it is assumed (sous entendu Fr) that it is Y and we also notice that it follows form the assumptions tht every singular zero of ϕ in non-singular relative to S. In this situation we can write down the following diagram of locally free (*sheaves*) of modules over V^1):

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \Omega^1_{X/Y}\otimes\mathcal{O}_{V'} & & & & \\
 & & & {}^{\mu}\nearrow & & & & & \\
0 & \longrightarrow & P^1_{X/Y}\otimes\mathcal{O}_{V'} & \longrightarrow & \Omega^1_{X/Y}\otimes\mathcal{O}_{V'} & \longrightarrow & \Omega^1 V^1/S & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & {}^{\nu}\nearrow & & \\
 & & \mathcal{O}_{V'} & \longrightarrow & \Omega^1_{X/Y}\otimes\mathcal{O}_{V'} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & L=\omega^{-2}_{X/Y}\otimes\mathcal{O}_{V'} & & 0 & & & &
\end{array}
$$

The columns come from the exact sequence of transitivity for the smooth morphisms $X\to Y$ and $Y\to S$ and tensoring with $\mathcal{O}_{V'}$ (this remains exact since all the modules in the sequence are locally free). The horizontal line is a particular case of an exact sequence obtained every time when over X over S we have a section ψ of a locally free module F and if we take the scheme of zeros W we find an exact sequence

$$F^v\otimes O_X\to\Omega^1_{X/S}\otimes O_W\to\Omega^1_{W/S}\to 0$$

and if X/S is smooth the first homomorphism is injective exactly at the point where W is smooth over X with a "good" relative dimension (i.e. everywhere in the present case). This exact sequence is an immediate consequence of the exact sequence

$$J/J^2\to\Omega^1_{X/S}\otimes\mathcal{O}_W\to\Omega^1_{X/S}\to 0$$

which appears in Par. 16 (we could state [mettre en corollaire] the version mentioned here).

The characterization of the set of points where we can set a zero on the left is contained is the Jacobian criterion.

Let us note that we have a canonical isomoprhism $P^1_{X/S} = \Omega^1_{X/S} + O_X$ hence $P^{V1}_{X/S} = G_{X/S} + O_X$ (in the original the G is elongated).[71] On the other hand, we verify that the composed homomorphism μ of the diagram 1 is zero on the factor $O_{V'}$ and on the factor $G_{X/S} \otimes \mathcal{O}_{V1}$ it reduces to the homomorphism $M(\phi)$ (illegible) deduced from the section $M(\phi)$ of Sym 2 $(\Omega^1_{X/S}) \otimes \mathcal{O}_{V'}$ already mentioned. Thus at point x of X, $M(\phi)$ is non-degenerate, i.e. $M(\phi)'$ is surjective if and only if M is surjective at x and we see that in the diagram 1 this is also equivalent to saying that V is surjective at x (since one and the other mean that the canonical homomoprhism of the sum of the two mentioned submodules of $\Omega^1_{X/S} \otimes \mathcal{O}_{V'}$ into the latter is surjective at x.)

We find therefore:

Proposition 7. *Under the preceding conditions (to be recalled) the underlying set of* $V(\phi)^{\text{sup sing}}$ *is nothing else but the set of points of* $V(\phi)^{\text{sing}}$ *where the morhism* $V(\phi)^{\text{sing}} \to Y$ *(of smooth preschemes over* S *of relative dimension* $m-1$ *and* m *respectively) is ramified.*

In the language of the fathers (en termes de papa Fr) (which we should give as a remark) a point $X \in V(\phi)$ is thus supersingular relative to Y if and onlyif "it consists of at least two coinciding (infinite near) singular points (confondus Fr)...

We may and we have to make precise Proposition 7 from the point of view of an identity of sub-preschemes and not just of subsets. Indeed, $V(\phi)^{\text{sup sing}}$ has been defined as a closed pre-subscheme of X or (Fr) we could equally well define a natual closed subscheme of V in such a way that the underlying subset should be the set of ramification points with respect to Y. Indeed it is enough to express the set of points where a certain homomorphism of locally free modules $Q = \Omega^1_{Y/S} \otimes \mathcal{O}_{V'} \to M(= \Omega^1_{V'/S})$ is not surjective. If q and r are their respective ranks this is also the set of points where $\Lambda^1 Q \to \Lambda\sqrt{M}$ is not surjective this is also the zero set of the evident section of $\text{Hom}(\Lambda^1 Q, \Lambda\sqrt{\ } M) \simeq (\approx)(\Lambda\sqrt{\ } Q) \otimes (\Lambda^r R) \otimes (\Lambda^1 Q)^v$, thus the underlying set of a closed sub-prescheme of zeros of this section, let us call it $\text{Ram}(V'/Y)$. I say that the latter subscheme is identical to $V(\phi)^{\text{sup sing}}$. This is a simple exercise about the diagram above, taking into account that $V(\phi)^{\text{sup sing}}$ is defined by the same procedure as the one made explicit for $Q \to R$ but in terms of the homomoprhism $P(= P^{v1}_{X/Y} \otimes \mathcal{O}_{V'}) \to S(= \Omega^1_{X/Y} \otimes \mathcal{O}_{V'})$ as follows from the description of μ given above. We are therefore reduced to the following general situation:

We have on a ringed space W a locally free module M of rank m and two locally free submodules P and Q of respective ranks p and q such that $p + 1 = m + 1$, we use the

[71]Is the elongated G the tangent sheaf? [Tr]

previous construction relative to morphisms $P \to M/W = S$ and $Q \to M/P = R$ to find the sections a) of

$$P \otimes \det S \otimes \det P^{-1} = P \otimes \det M \otimes \det P^{-1} \otimes \det Q^{-1}$$

and b) of

$$= Q \otimes \det M \otimes \det P^{-1} \otimes \det Q^{-1}$$

which we may also consider as homomorphisms of $L = \det P \otimes \det Q \otimes \det M$ into P respectively Q. (Nota bene: we denote for a locally free module F by $\det F$ its highest exterior power and we use the fact that for a short exact sequence

$$0 \to F^1 \to F \to F^{11} \to 0$$

of such modules we hae a canonical isomoprhism

$$\det F = \det F^1 \otimes \det F^{11},$$

This being given [Fr], we have the *commutativity of the diagram*

$$\begin{array}{ccc} P & \longrightarrow & M \\ {\scriptstyle a}\uparrow & & \uparrow \\ \det P \otimes \det Q \otimes \det M^{-1} = L & \xrightarrow{b} & Q \end{array}$$

(possibly up to sign, this depends perhaps on the conventioins adopted to define some of the canonical isomorphisms used, which govern the choice of the sign. . .)

Hence the ideals $V(a)$ and $V(b)$ are identical since P and Q are locally direct summands in M. Nota bene at the points $\notin V(a) = V(b)$, L is exactly the intersection of P and Q in M.

It remains only to completely clarify the particularly described situation with $X/Y/S$ and ϕ described in the diagram D above to make explicit a and b. We find first of all

$$\begin{aligned} \det P &= (\omega_{X/Y})^{-1} \otimes \mathcal{O}_{V'} \\ \det Q &= (\omega_{Y/S}) \otimes \mathcal{O}_{V'} \\ \det M &= \omega_{X/S} \otimes \mathcal{O}_{V'} = \omega_{X/Y} \otimes \omega_{Y/S} \otimes \mathcal{O}_{V'} \end{aligned}$$

(the last isomoprhism coming from the exact sequence of transitivity for the Ω^1 for smooth morphisms

$$X \to Y \text{ and } Y \to S)$$

hence

$$L = \omega_{X/Y}^{-2} \otimes \mathcal{O}_{V'}$$

(Nota bene: $\omega_{X/Y}$ denotes det $\Omega^1_{X/Y}$). *This says that the homomorphism* a) *factors through the factor* $\mathcal{O}_{V'}$ of $P^v_{X/Y} \otimes \mathcal{O}_{V'}$ and en tant [Fr] as the homomorphism of this factor it is determined precisely by the section $D =$ det $M(\phi)$ already mentioned elsewhere (always modulo a sign to be determined). Of course the homomorphism b) is deduced taking into account the commutativity announced in the lemma.

The non-supersingular points are therefore exactly the ones where $W^{-2}_{X/Y} \otimes \mathcal{O}_{V'} \to \mathcal{O}_{V'}$ is an isomorphism and at these points the factor $O_{V'}$ can be identified exactly with the intersection of $P^1_{X/Y} \otimes \mathcal{O}_{V'}$ and $\Omega^1_{Y/S} \otimes \mathcal{O}_{V'}$ in $\Omega^1_{X/S} \otimes \mathcal{O}_{V'}$. Thus over the open subset of V' consisting of non-supersingular points we have the exact sequence,

$$0 \to \mathcal{O}_{V'} \to \Omega^1_{Y/S} \times \mathcal{O}_{V'} \to \Omega^1_{V'/s} \to 0$$

therefore also the exact sequence

$$0 \to \mathcal{O}_{V'} \to \Omega^1_{X/S} \otimes \mathcal{O}_{V'} \to (\Omega^1_{X/Y} + \Omega^1_{Y/S}) \otimes \mathcal{O}_{V'} \to 0$$

It is necessary to summarize the whole situation in a recaptulating proposition and in a slightly more general form by taking a section ϕ of an *invertible module* K over X which obligates us to twist by K a certain number of modules intoduced in the previous considerations. This is indeed the situation encountered in the pragraph about hyperplane sections. Let us also remark that $V(\phi)^{\text{sing}}$ and $V(\phi)^{\text{sup sing}}$ do not change if we replace ϕ by $u\phi$ where u is a unit, indeed $d\phi/V(\phi)$ and $M(\phi)$ are multiplied by$u$ and $D(\phi)$ is multiplied by u^M ($M=$ relative dimension of X over Y).

Remark. With the mentioned conditions about X, Y, S, ϕ since $U = V(\phi)^{\text{sing}} - V(\phi)^{\text{sup sing}}$ is unramified over Y its fibers over Y are discrete which implies that for every $y \in Y$, every ordinary singular zero of ϕ_y in X_y is isolated in the set of singular zeros, i.e. it is isolated in the set of non-smooth points of $V(\phi_y)/k(y)$.

We conclude [Fr] p.c.x immediately that if $d \geq 2$ it is a geometrically normal point of V_y over $k(y)$ (since in any case V_y is Cohen-Macauley).

We may prove more generally under the conditions of Definition 1 that x is an odinary singular zero thus x is an isolated singularity of $V(\phi)$ in the sense that every generalization of x^1 of x in $V(\phi)$ is a regular point of $V(\phi)$ (thus if the singular locus of $V(\phi)$ is closed, for example if X is "excellent" x is also an isolated point of the singular locus). (For good measure, really we should include this result in Ch. IV but Par. 16 does not at all seemto be the right place. Where do we put it?).

It follows that if X is smooth ovr a field k andif x is a geometrically ordinary singular zero then x is an isolated point of the set of points of non-smoothness of $V(\phi)$, thus it is geometrically normal if $\dim_x(X) \geq 2$, (but necessarily non-normal if $\dim_x X = 1$ as one sees, for example, in the case of the type $\phi = xy$, x and y coordinate functions in the affine plane).

Here is a diagram analogous to (D) but for a section ε of an invertible module K.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \Omega^1_{X/Y} \otimes \mathcal{O}_{V'} & & & & \\
 & & & {}^{\mu}\nearrow & \uparrow & & & & \\
0 & \longrightarrow & P^1_{X/Y}(K)^v \otimes \mathcal{O}_{V'} & \longrightarrow & \Omega^1_{X/S} \otimes \mathcal{O}_{V'} & \longrightarrow & \Omega^1_{V'/S} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & \nearrow_{\nu} & & & \\
 & & K^{-1} \otimes \mathcal{O}_{V'} & \longrightarrow & \Omega^1_{Y/S} \otimes \mathcal{O}_{V'} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \omega^{-2}_{X/Y} \otimes K^{-1} \otimes \mathcal{O}_{V'} & & 0 & & & &
\end{array}
$$

[This part was crossed out by Grothendieck, the translator is not sure why. It is reproduced here for completeness where the crossed out part ends we should say *no longer crossed out*.]

Upon arriving at this point , admirable for its generality, I realized also that these developments are still too special and that in place of a prescheme defined by a single equation section of O_X it wold be proper to consider one defined by m-sections or what is the same by a homomorphism $E \to O_X$ where E is a locally free module of rank m.

I leave it up to you [Dievdonne] the case of deciding if it is proper s'il a lieu [Fr] from the start to begin with the general case, following the principle that one should proceed from general to special.

Let us introduce from the start a terminology for a ring A with augmentation $A \to A/I$ which will be used also in the sheaf context. The augmentation is called *regular* (or shall we say quasi-regular with definitions adopted in Par. 19?) If

a) I/I^2 is an A/I-module projective and of finite type, and
b) The homomorhism $O\colon \mathrm{Sym}(I/I^2) \to \mathrm{gr}_I(A)$ is an isomoprhism.

We will say that the augmentation is of the type "ordinary quadratic" if

a) The above is verified and also
b) The canonical preceding homomorphism has a kernel generated by O^2, hence the kernel K is invertible sub-module "non-degenerate" of
$\mathrm{Sym}^2(I/I^2)$.

By non-degenerate we mean that the corresponding homomorphism $K \otimes (I/I^2)^v \to I/I^2$ deduced by using the canonical homomorphism $\mathrm{Sym}^2(I/I^2) \to \mathrm{Hom}((I/I^2)^v, I/I^2)$ is bijective, or what is the same, surjective (since we are dealing with projective modules over A/I of the same rank at every point) also the quadratic form corresponding over $(I/I^2)^v$ is non-degenerate. Anyway, it suffices to verify the latter conditions for the reduced fibers en [Fr] the different prime ideals (or even only at maximal ideals) of A/I.

Proposition 8. *For the augmentation $A \to A/I$ to be of ordinary quadratic type, it is necessary and sufficient that the following conditions should be verified:*

1^o *The $\mathrm{gr}_I^n(A)$ are projective and of finite type over A/I,*

2^o *If we denote the rank of $\mathrm{gr}_I^1(A) - I/I^2$ and a $p \in \mathrm{Spec}(A/I)$, then the rank of $\mathrm{gr}_I^n(A)$ at the same point is equal (or simply $\geq$) to $\binom{n+d}{d} - \binom{n-2+d}{d}$,*

3^o 1^o *and* 2^o *imply already that $K = \mathrm{Ker}\, O^2$ is an invertible module locally a direct factor in $\mathrm{Sym}^2(I/I^2)$. The submodule $K = \mathrm{Ker}\, O^2$ of $\mathrm{Sym}^2(I/I^2)$ is non-degenerate.*

The proof is immediate by starting from the following lemma:

Lemma 8.1. *Let B be a ring, M a projective B-module of finite type, $\phi \in \mathrm{Sym}^2(M)$ a 'non-degenerate' element, then the multiplication by $\phi\, \mathrm{Sym}^2(M) \to \mathrm{Sym}^B(M)$ [α and B illegible but look like n, $n+2$ or $n+2, n$ Tr] is an isomoirphism over a submodule direct factor, thus the cokernel is a projective module of rank $\genfrac{}{}{0pt}{}{n+3}{d} - \genfrac{}{}{0pt}{}{n-2+d}{d}$ if M has rank d.*

We are immediately reduced to the case where B is the spectrum of a field or is trivial (in fact the "non-degenerate" only serves us to assure that the $\phi(X)$, $x \in \mathrm{Spec}(B)$ are all different from zero and we could formulate the lemma more reasonably by replacing two by an integer $k \geq 1$ arbitrary without ralking about the non-degeneracy).

Proposition 9. *Let A be a ring augmented into A/I with a regular augmentation, I/I^2 being projective of rank n over A/I. Let E be an A-module projective of rank m $u\colon E \to I$ a homomorphism hence the homomorphism $u \otimes \mathrm{id}_{A/I}\colon E \otimes_{A/I} \to I/I^2$. Let $B = A/u(E)A$ with the augmentation $B \to B/J = A/I$ where $J = I/u(E)A$. The following two conditions are equivalent:*

(i) The ring B has an augmentation $B \to B/J$ of ordinary quadratic type and of rank $n-(m-1) = n-m+1$. (The rank of the augmentation is by definition the one of the projective module J/J^2).

It would be perhaps clearer to introduce the notation $V = I/I^2$ and $W = J/J^2 = V/M\,E \otimes A/I$. the homomorphism qualified as natural in (ii) is obtained by noting that K is mapped in every case | to (dons [Fr]) $I^2/(I^3 + u(IE)) = (I^2/I^3)\mathrm{Im}(IE)$.

(ii) $K = \mathrm{Ker}(u \otimes \mathrm{id}_{A/I})$ is an invertible module and the natural homomorphism

$$K \to \mathit{Sym}^2(J/J^2) = \mathit{Sym}^2((I/I^2)/\mathrm{Im}\, E \otimes A/I)$$

is non-degenerate.

$$\mathit{Sym}^2(I/I^2)/\mathrm{Im}(I/I^2) \otimes E \otimes A$$

where

$$(I/I^2) \otimes (E \otimes A/I) \to \mathit{Sym}^2(I/I^2)$$

is defined via the canonical homomorhism of multiplication

$$I/I^2 \otimes I/I^2 \to \mathit{Sym}^2(I/I^2).$$

With these made explicit the proof of Proposition 9 is almost trivial.

Corollary 10. *Let A be a local ring of dimension n, m its maximal ideal, $k = A/M$, E a free A module of rank in $u\colon E \to M$ a homomorphism, $B = A/u(E)A$, $n = M/u(E)A$ the maximal ideal of B, $V = M/M^2$, $W = N/N^2 = V/M(E \otimes K)$. The following conditions are equivalent:*

(i) B has the dimension $(n-m)$ (i.e. in terms of a basis (ϕ_i) of E the images of ϕ_i in A form an A-regular sequence and the closed point of $\mathrm{Spec}\, B$ is an ordinary quadratic singularity) by which we mean that the augmentation $B \to B/N$ is of the ordinary quadratic type.

(ii) There exists a basis $\phi_1, \phi_2, \dots, \phi_m$ of E such that

a) $A \Big/ \sum_{i=1}^{M-1} U(\phi_i) = A\,m-1$ is regular of dimension $n-(m-1)$, and

b) the image of $u(\phi_m)$ in $A\,m-1$ admits a closed point of $\mathrm{Spec}(A\,m-1)$ as an ordinary singular zero (if def. 1).

(ii)bis The rank of $M(E \otimes k \to V)$ is $(m-1)$ and for every base (ϕ_i) $1 \le i \le m$ of E such that the images of the ϕ_i $(1 \le i \le m-1)$ in V are linearly independent the conditions a) and b) of (ii) are satisfied.

(iii) The kernel K of $E \otimes k \to V$ is of dimension one and if $\phi \in K - \{0\}$ then the element of $Sym^2(W)$ defined by ϕ is a non-degenerate quadratic form.

[Here ends the crossed out part of Translator's note. No longer crossed out.

Let us note that the analogue of conditions (ii) and (ii)bis could already have been included (remonter Fr) in Proposition 9. It is proper to introduce the following terminology: If X is a regular prescheme, E a locally free module over X, $u: E \to O_X$ a homomorphism, i.e. u a section of E^v we say that the element X of $V(u) = V(u(E)O_X)$ is a "non-singular zero" for u if u is O_X regular, i.e. dim $O_{V(u),X} =$ dim $O_{X,X} - \text{rank}_x E$, and if in addition x is a regular point of $V(u)$, which in terms of a basis $\phi_1, \phi_2, \ldots, \phi_m$ of E in a neighborhood of x is expressed by saying that the $u(\phi_i)$ form a part of a regular system of parameters of $X_{X,x}$, in the contrary case x is called a singular zero.

We say that x is an ordinary singular zero if it satisfies the equivalent conditions of Corollary 10 (which implies that it is also a singular zero), where evidently $A = O_{X,x}$, etc. We say that x is a supersingular zero if it is singular without being ordinary singular. Thus not supersingular = non-singular or ordinary singular.

Finally, if X is smooth over a field K we introduce the geometric variants of these notations in an evident way. We note that in these notions mis a' part la condition dimensionelle [Fr] = the condition of O_X regularity, the character of x is seen indeed in the nature of the point x in $V(u)$, i.e. in the local ring $O_{V(u),x}$ (possibly considered as an algebra over K). If (in the case of a ground field K) x is rational over K, the relative notion over K coincides with the absolute notion as we see in the case of an invertible E. Finally, if X is smooth over a prescheme S and $u: E \to O_X$ is again a homomorphism with E locally free of finite type, we define the corresponding notions relative to S by considering fiber by fiber. We will note that if x is non-supersingular relativ eto S then $V(u)$ is that over S at x (since u is an O_X-regular relative to $S \ldots$) [TR REMARK: S is altered and could be a Y in the above few lines.] Of course (to be stated at least as a remark after the definitions in their final form) we may also just as well start from a section u of a locally free module E, the preceding definitions apply to interpret u as a homomorphism $E^v \to O_X$.

If X is smooth over Y, $u: E \to O_X$ is given with E locally free of rank in then the set of zeros of u singular relative to S [or is it Y, Tr. questions] is the underlying set of a subscheme $V(u)^{\text{sing}} = V^1$ well defined of $V(u) = V^0$ obtained thus: we consider d^1u a section of $P^1_{X/Y}(E^v)$ its restriction to V^0 may then be interpreted as a homomorphism $E \otimes O_{V^0} \to \Omega^1_{X/Y} \otimes O_{V^0}$ and $V(u)^{\text{sing}} = V^1$ denotes the prescheme of zeros of the corresponding homomorphism $E \otimes O_{V^0} \to \Omega^m_{X/Y} \otimes O_{V^0}$, i.e. a well defined section of

$\Omega^m_{X/Y} \otimes \det(E)^{-1} \otimes O_{V^0}$ to characterize using differential method the ordinary singular zeros relative to Y we note that it is necessary simply to express the conditions of Proposition 9 for the homomorphism deduced from u. $P^\infty_{X/Y}(E)_{V^1} \to (P^\infty_{X/Y})_{V^1}$ which is in addition reduced to an "order two" condition dealing only with the corresponding homomorphism $P^2_{X/Y}(E)_{V^1} \to (P^2_{X/Y})_{V^1}$. It is necessary to express in the first place that the kernel K of $E_{V^1} \to (\Omega^1_{X/Y})_{V^1}$ at x is invertible and a direct factor, i.e. that the described homomorphism (which, because of $x \in V^1$, is of rank $(m-1)$ at x) is exactlyof rank $(m-1)$ at x. (Nota bene: *the rank* of a homomorphism of locally free modules at a point is by definition the rank of the homomorphism of vector spaces over $K(x)$ that correspond to it) which can be expressed by considering $\Lambda^{m-1} E_{V^1} \to \Omega^{m-1}_{X/Y}{}^{V^1}$ the scheme V'' of zeros of this homomorphism $(V'' V')$ and by writing simply$x \in V''$. The definition of V'' insures that $K/V' - V''$ is an invertible module and tht the cokernel W of $(*)$ is such that $W/V' - V''$ is locally free of rank equal to $d-(m-1)$ (where d is the relative dimension of X over S equal to the rank of $\Omega^1_{X/Y}$ [Translator: the previous S ought to be a Y I am sure]). Applying the construction made explicit in Proposition 9, we find a homomorphism $M \colon K \to \mathrm{Symm}^2(W)$ and it remains only to express that the latter is "non-degenerate" at x. For this let us introduce the discriminant $D \in \lceil(V' - V'', K^{-n+m-1} \otimes \det(W)^2)$ (where the exponents denote the tensor powers of invertible modules). Setting $V^2 = V(D)$, which is a closed subscheme of $V' - V''$, we find therefore that the set of ordinary singular zeros relative to Y is nothing else but $V' - V'' - V^2$, i.e. the set of supersingular zeros relative to S (or Y Tr.) is $V''\ V^2$ (should it be $V'\ V^2$ translator's question).

At the moment of introducing V'' in asserting that W and K are locally free over $V' - V''$, we have used a general fact that I express here in a lemma (where $V' - V''$ est devauu [Fr] becomes X):

Lemma. *Let $V \colon E \to F$ be a homomorphism of locally free modules over a prescheme X, m an integer > 0, let us suppose that $\Lambda^m u$ is zero, then $\Lambda^{m-1} u$ is not zero at any point (from the point of view of reduced fibers bien sur [Fr]) if and only if $u(E)$ is a sub-module locally a direct factor of F, locally free of rank $(m-1)$ or if and only if u is of rank $(m-1)$ at every [illegible] point of X.*

Il ya lieu [Fr]. It is proper to summarize these constructions in a proposition that generalizes Proposition 6.

It is necessary to see how we can generalize the same Proposition 7. To do this we suppose that Y itself is smooth over a prescheme S of relative dimension N, thus X is smooth over S of relative dimension $N + d$, where d is the relative dimension of Y. We suppose also with the above notations $V'' = \emptyset$, i.e. W locally free of rank $d - m + 1$ (this is always the case if $m = f1!$). Finally, we suppose that the closed subschemes V^0 and

V' of X are smooth over S, of reltive dimension minimal at every point (for the given d, N, m), that is to say as we see immediately of relative dimensions $(N+d)-m$ and $(N+d)-m+-(d-m+1)=N-1$. (From the point of view of writing up the last seem to pose brutally these numbers, perhaps rather in terms of fiber by fiber codimension, thta of V^0 in X being m, that of V' in V^0 being $(d-m+1)$ and to state as a remark the justification of this choice by the principle of minimal generic dimension.

If $E \to F$ is a homomorphism of locally free modules over the regular prescheme X of ranks m and d ($m \leq d$) and if Z is the set of points of X where the rank of u is $\leq (m-1)$, then we can show that Z is of codimension $\geq d-m+1$ at every one of its points.

This done, it would be necessary to verify that the set V^2 of supersingular points of u relative to Y is nothing else but the set of points where $V^{(1)} \to Y$ is ramified and to make this point more precise as an identity of closed subschemes of X, $V^2 =$ "sub-prescheme of ramification" of $V^1 \to Y$. I have not done this exercise in detail, but no doubt it can be done by essentially the same kind of devissage as the one developed after Propositon 7.

EGA V – Section 17

IV.17.15 Smooth Forms (illegible) and elementary singularities illegible New EGA ???

Nota Bene: I have just noticed tht the terminology introduced in my formulation $S \cup P$ of supersingularity is unreasonable and it conflicts in particular with the recent terminology. In any way, you must have noticed that in Def. 1c – of the text it should read "degenerate" in place of "non-degenerate" La canular [Fr] (The hoax (Ecole Normale Superieur lingo) Tr) is given by varieties of even dimension in characteristic two defined, for example, by an equation $\phi = 0$ in an ambient variety of odd dimension, since with the terminology of my notes such a variety cannot have an "ordinary singular point" (= ordinary quadratic singularity), i.e. ϕ cannot have an "ordinary singular zero" due to the fct that in characteristic two a quadratic form in an odd number of variables is always "degenerate". But the whole world has always considered that even in characteristic two the origin of the affine cone $X^2 + YZ = 0$ is an ordinary quadratic singularity. In the present notes I give the notion of a smooth quadratic form (or "ordinary") (over a locally free module of finite type), in such a way that non-degenerate $\Rightarrow$ smooth, the reciprocal being true if E is of even rank or the residual characteristics of the base S are all different from two. This notion having been introduced, I propose to preserve the terminology "supersingular" (which does not conflict with any recent terminology) opf $S \cup P$, which corresponds to the notion of a non-degenerate quadratic form; we will also speak about a supersingular point of a locally noetherian prescheme, where geometrically supersingular for a prescheme locally of finite type over a field, in the same reuse (by regarding the homomorphism $\mathrm{Sym}(M/m^2) \to \mathrm{gr}_m(A)$ and by saying that the kernel cannot be generated by a quadratic non-degenerate form, thus that which is $S \cup P$ is called wrongly [Fr] "ordinary singular zero" is called "singular non-supersingular zero" or "singular quadratic non-degenerate zero" if we tient [Fr] a terminology close to the recent terminology in characteristic zero. Also in the same way we could speak about "non-degenerate quadratic singularity" quand on est [Fr] locally of finite type over a field K.

By contrast the terminology "ordinary singular zero" a mieux [Fr] "ordinary quadratic zero" and also "ordinary quadratic singularity", "singularity geometrically ordinary quadratic" can be extended (conforming to the usage) as corresponding to the notion of a smooth quadratic form. In addition, it may be indicated to replace the word "ordinary" by the word "elementary" and to extend this terminology to singularities not necessarily quadratic but of any multiplicity. Tell me (Dieudonne) your impression with regard to this. It seems bien [Fr] on the other hand that the text $S \cup P$ is formally correct, in particular in Proposition 7 the notion that is really being introduced is that of a supersingular zero.

This does not contradict that it would be proper to at least as a remark to introduce also the subscheme $V(\phi)^{\text{ult sing}}$ of $V(\phi)^{\text{sup sing}}$ corresponding to considering ultra-singular zeros, i.e. those that are singular without being elementary quadratic; they will be described essentially by the same procedure as $V(\phi)^{\text{sup sing}}$ by taking the prescheme of zeros fo the "corrected discriminant" introduced later in place of the ordinary discriminant. Thus the only corrections to $S \cup P$ seem to be of terminological order (and they have to porter [Fr] equally, of course, to the terminology introduced in $S \cup P$ page 10) Pr contue [Fr] (in contrast).

IV 23.9.2 is false as such except if characteristic K is not two or the irreducible components of X even dimension; in the general case it is necessary to suppress "fn satisfies the equivalent conditions of 8.8 in particular"; the rest of the proposition seems correct, and one should be able to prove it in a manner quite analogous by showing that for given $x \in X(k)$ there is a hypersurface H of given degree ≥ 2 which is tangent to X at x and such that x is an ordinary quadratic points of $X \cap H$, of dimension one less… Bien meltre les pieds dons le plat [Fr], by remarking tht if k *is* of characteristic two and X is connected non-empty of odd dimension then the conditions of 8.8 are not verified, i.e. L/K is necessarily inseparable (I believe of degree two exactly if X P^r, but without guarantee…) Let S be a prescheme, E a model locally free of finite type over S, ϕ a section of $\mathrm{Sym}^n(E)$, i.e. an "n-form over E^v" where $n \geq 0$.

To give ϕ is equivalent to giving a section of $O_P(n)$ over $P = P(E)$ (ref. to III) and defines therefore a subscheme $V(\phi)$ of P. We say tht the form ϕ is *smooth* if $V(\phi)$ is smooth and if in addition for every $x \in S$ $V(\phi)_s \neq P_s$ (i.e. $\phi(s) \in \mathrm{Sym}^n_{k(s)}\ E \otimes k(s)$ is not zero). We see immeditely that ϕ is smooth if and only if for every $s \in S$ $\phi(s)$ is smooth (which reduces us to the spectrum of a field) and that the notion of a smooth form is invriant under the change of ground field (which reduces us to the case of an algebraically closed field); we can summarize these two properties by saying that if $S^1 \to S$ is a surjective morphism then ϕ is smooth if and only if its inverse image ϕ^1 over S (S^1 illegible) is such. Of course if E is a projective module of finite type over a ring A and $\phi \in \mathrm{Sym}^n_A(E)$ we say that ϕ is smooth if ... Since $E = A^{N=1}$ to give ϕ is equivalent to giving a polynomial homogeneous of degree n in the variables $X_0, X_1, \dots, X_r$ and the jacobian criterion implies immediately that [illegible] ϕ is smooth if and only if the ideal generated by ϕ and the ϕ_{X_i} contains a power of the augmentation ideal $(X_0, \dots, X_r)$, i.e. contains a power of each variable X_i. More precisely the subscheme of P^r defined by the homogeneous ideal defined by ϕ and the ϕ_{X_i} is exactly formed by the points of $V(\phi)$ at which $V(\phi)$ is not smooth over $\mathrm{Spec}(A)$ with a relative dimension $r - 1$.

15.2. Let X be a prescheme, J an ideal quasi-coherent over X, we say that the homomorphism "d'augmentation" $O_X \to O_{X/J}$ is an "elementary augmentation of multiplicity n" if it satisfies the following conditions:

a) $J/J^2 = N_{X/Y}$ is locally free of finite type over $Y = V(J)$

b) The kernel of the canonical homomorphism $\Psi\colon \mathrm{Sym}_{O_Y}(N) \to \mathrm{gr}_J(O_X)$ is generated by the kernel K of Ψ^n, which is an invertible submodule locally a direct factor of $\mathrm{Sym}^n_{O_Y}(N)$ (i.e. it is locally generated [generatable] [engendrable] [Fr] over Y by a section ϕ which is not zero at any point.

c) Ladite [Fr] ϕ (which is defined locally only up to multiplication by a unit) is a *smooth* form.

(Nota bene: We could have introduced in 15.1 the notion of a submodule invertible and smooth of $\mathrm{Sym}^n(E)$, to tell the truth geometrically more important than that of a smooth *section*, since $V(\phi)$ is in fact defined by such a submodule. . . , this would have the advantage to allow b) and c) as a single condition.)

If $n = 2$ we talk about "augmentation of elementary quadratic type". If A is a ring with an augmentation $A \to A/J$ we agree again to say tht this augmentation is elementary of multiplicity n, if it is such for ... If A is a local ring we will say (on dira [Fr]) simply by abuse of language that A is "elementary of multiplicity n" if the augmentation $A \to A/r(A)$ is elementary of multiplicity n; note that this implies $n \geq 2$ and if A is noetherian A is necessarily not regular. If X is a prescheme, n an integer ≥ 2, we say that $x \in X$ is an elementary singular point of multiplicity n if its local ring $O_{X,x}$ is elementary of multiplicity n. (Let us remark that this terminology is in agreement with the general notion of multiplicity due to Samuel). For $n = 2$ we will speak in particular about elementary quadratic singularity (or ordinary, in the classical terminology). We introduce also conforming to the general usage the "geometric" variants for X locally of finite type over a field: $x \in X$ is called a geometrically elementary singularity of multiplicity n if for every (or what is the same for one) extension K of k and every (or one) point z of X_K over x, rational over K, z is an elementary singularity of multiplicity n.

15.3. "Generalize" $X \cup P$ Proposition 8 to the case of multiplicity n.

15.4. "Generalize" $S \cup P$ Proposition 9 and Corollary 10 to the case of multiplicity n. (The idea is not to generalize but to give variants).

15.5. To introduce the notion of an elementary singular zero of multiplicity n (for $n = 2$ a singular quadratic elementary zero) of a section ϕ of O_X, or more generally of a module locally free ovr an X locally noetherian and the corresponding geometric notion (over a base field K).

15.6. Let X be a subscheme of a locally noetherian regular X' and let $x \in X$. In order that x should be an elementary singularity of multiplicity n of X it is necessary and sufficient that there should exist anopen neighborhood U of x and an O_u regular sequence $\phi_1, \phi_2, \ldots, \phi_d$ such that $X \mid U = V(\phi_1, \ldots, \phi_d)$ and such taht x should be an elementary singular zero of multiplicity n of $(\phi_1, \ldots)$. In particular if A is noetherian local ring which is elementary of multiplicity n, then A is a "complete intersection" ring also we see due to 15.4 that we can find a neighborhood of x a sub-prescheme X'' of x regular containing X and such that X can be described in X'' by a single equation $\phi = 0$ admitting x as an elementary singular zero of multiplicity n.

We conclude in particular tht if A is a local noetherian elementary of multiplicity n then A is Cohen-Macauley. We prove in Paragraph 20 (by an easy blow up calculation) that the closed point of $\operatorname{Spec} A$ is the only singular point of $\operatorname{Spec} A$, it follows that A is normal if and ohnly if $\dim A \geq 2$, reduced if and only if $\dim A \geq 1$. If $\dim A = 0$, then A is elementary of multiplicity n if and only if M/M^2 is of rank 1 and n is the smallest integer such that $M^n = 0$; we note that such rings are also the quotients of discrete valuation rings by the n-th power of their maximal ideal.

15.7. This is the place to state the "geometric" variants of 15.6, we find in particular that if X is a prescheme locally of finite type over a field k, $x \in X$ and if $n = \dim_{x}X$ then x is an elementary singularity of multiplicity n of X if and only if there exists an open neighborhood U of x such that U can be embedded as a sub-prescheme of a smooth prescheme X' over K connected and of dimension $n+1$ (???) [illegible] defined by an equation $\phi = 0$, x being a geometrically ordinary singular zero of multiplicity n of ϕ. We shall say also in 15.5 on aura dit [Fr] (we have said) that this means that the value at x of the principal part $d^n\phi$ of ϕ (which a ????? is an element of $P^N_{X'/K} \otimes k(X)$ [illegible] and more precisely in its augmentation ideal I) is in fact an element of $I^n = \mathrm{Sym}^n_{O_u}(\Omega^1_{X'/k} \otimes k(x))$ and because of this is a smooth form [NB: Recall in 15.1 the set of points of smoothness of a form is open]. We remark also that such a point is isolated in the set of non-smooth points, it is geometrically normal (resp. geometrically reduced) if and only if $d \geq 2$ (resp. $d \geq 1$). If $d = 0$ and $k(x) = k$, (i.e. x is isolated and x k-rational) then x an elementary singular point of mult n means that $O_{X,x}$ is k isomorphic to $k[T]/(T^n)$.

If $k = 1$ x rational over k the notion of an elementary singular point of multiplicity n corresponds in the classical terminology to "point ordinary singular with n distinct tangents" (We have already made explicit (on aura deja explicite [Fr] in 15.2 that if x is a point of X rational over k then the notion of elementary singularity of multiplicity n in the absolute or relative sense is the same, this remains also true if $k(x)$ is a finite separable extension of k – without a doubt in addition "finite" is not used and this fact deserves to be inserted as a corollary or as a proposition). Nota Bene: I have included in Paragraph 20 a section (un No. [Fr]). I had included about blowing up a prescheme X along a closed sub-prescheme Y such that the augmentation $O_X \to O_Y$ is elementary of multiplicity n (to which I allude in 15.6). I remark that what is involved here is a short section and welcome whose only ingredients are the general smoothness results of Paragraph 17 and the definition at the start of 17.15.2 (and even we could avoid the latter). There would be no harm to incorporate these results from here, for example immeditaely after the definition 15.1 above; then la gamme [Fr] "geometric" of 15.2 and the continuation 15.3 and 15.7 could be separated from these results by grouping them in a No. 17.16.

Anyway, I noticed during the writing tht the new foreseen No. 15 about smooth forms will blow up into at least two sections quite distinct and independent, one containing the general 'sorites' for the stuff "elementary of multiplicity n", any n the other containing the (a) characterization of smooth *quadratic* forms and which does not borrow from the preceding except in No. 15.1, i.e. practically the definition of the smoothness of a form.

Appendix 17.16

Smooth Quadratic Forms

16.1. Let Q be a section of $\mathrm{Sym}^2(E)$ (E locally free of finite rank) such that Q (which one can also interpret as a quadratic form over E^v) defines a bilinear symmetric form B over E^v, hence a homomorphism $E^v \to E$ and by passing to determinant modules one finds det $E^v \to$ det E hence finally a section $d(Q) \in \lceil(\det(E)^{\otimes 2})$ called (up to an error)[72] *discriminant of the quadratic form* Q. If $E = O_S^n$ this is simply the section of O_S which is the determinant of the n by n matrix with coefficients in O_S which defines (expressed) Q.

In all tht follows[73] let us next notice:

Proposition 16.2. *Let us suppose that for every $s \in S$, we have characteristic of $k(s) \neq 2$ or that the rank of $E \otimes k(s)$ is even.*

Then in order for Q to be smooth it is necessary and sufficient that Q should be "non-degenerate", i.e. that $d(Q)$ should be invertible. (Anyway the condition is sufficient without any restrictions on S or E). This last assertion is given one only for convenient reference later, since it is trivially contained in the first one, the hypothesis that Q is non-degenerate implies in effect that E is of even rank at every point where the characteristic is 2. Besides, this one can prove directly without consideration of characteristic (by putting yourself over an algebraically closed field and by taking a basis) it expresses tht the $\frac{\partial Q}{\partial X_i}$ have no common non-trivial zeros, à fortiori they do not have a common non-trivial zero with Q. If the characteristic is not two the converse is true since by virtue of the formula x

$$2Q = \sum X_i \frac{\partial Q}{\partial X_i}$$

we see tht every common zero of $\frac{\partial Q}{\partial X_i}$ is also a zero of Q.

Finally, if characteristic $k = 2$ then the biliner form B associted with Q (defined by the matrix of the $\frac{\partial Q}{\partial X_i}$ is alternating, so that its "kernel" $N \subset E^v$ is such that E^v/N is of even rank, so that if E is itself of even rank this is also true about N. Consequently, if $N \neq 0$ the rank of N is at least two, hence it follows that there exists at least one non-trivial zero of Q over N, i.e. Q is not smooth.

N.B. This recourse to coordinates is decidedly offensive for us; we need it only to prove simply the

[72]sauf erreur
[73]taut desuite

Lemma 16.3. *Let E be a vector bundle of finite rank over a field k, $Q \in Sym^2(E)$, x a non-zero element of E^v, x' its image in $P(E)$, let us assume that $Q(x,x) = 0$, i.e. $x' \in V(Q)$. Then for $V(Q)$ to be smooth at x' and of dimension (rank $E - 1$) (i.d. $Q \neq 0$) it is necessary and sufficient that x' shuld not belong to the kenrel of the homomorphism $E^v \to E$ defined by the bilinear symmetric form associated to Q.*

16.4. The study that follows is designed essentially to give a critrion of smoothness for a quadratic form in the case not covered by 16.2, i.e. essentially for the case of a vector bundle of odd dimension over a field of characteristic two.

In this case *every* quadratic form Q over E^v is degenerate but one sees easily (by taking notably the "standard form") that it can still be smooth.

Let us introduce for every integer $n > 0$ "the standard quadratic form" Q_n over Z^n as a form with integer coefficients in the n variables, $X_1, \ldots, X_n$ and let us distinguish the two cases.

a)

$$n = 2m, Q_{2m}(X_1, \ldots, X_{2m}) = X_1X_2 + X_3X_4 + \cdots$$

b)

$$n = 2m+1, Q_{2m} + 1(X_1, \ldots, X_{2m+1}) = Q_{2m}(X_1, \ldots, X_{2m}) + X_{2m+1}$$

Lemma 16.5. *Let n be an odd integer let us consider $Q(X_1, X_2, \ldots, X_n) = \sum_i a_iX_i^2 + \sum_{i<j} b_{ij}X_iX_j$ a quadratic form in variables x_i with indeterminate coefficients a_i, b_{ij} such that the discriminant $d(Q)$ is a polynomial with integer coefficients and the q_i, b_{ij}. Then the "content" of this polynomial is equal to two (2), i.e. the greatest common divisor of its coefficients is two.*

Let us prove first of all that the content is a multiple of 2. Indeed, this means that if we specialize the polynomial over (en) the field $Z/2z$ it is identically zero which results from the fact that a quadratic form of odd degree over a field of characteristic 2 is always degenerate, i.e. has disciminant equal of zero. In order to prove that the content is exactly two it suffices to compute the discriminant of the standard form of degree n, on the one side it has to be a multiple of c on the other hand the calculation gives 2 (since it is one for even degree and two for the degree one...) It makes sense therefore to introduce the polynomial $\tilde{d}(Q) = (1/2)d(Q)$ in the coefficients of Q which again has integer coefficients so tht it takes a well defined value if we specialize the q_i, b_{ij} to any ring A, i.e. if one takes a quadratic form of degree n with coefficients in A.

We call $\tilde{d}(Q)$ the "*discriminant adjusted polynomial of the indeterminant quadratic form Q in n-variables*". And its value relative to the coefficients of a quadratic form q with coefficients in any ring A will be called "*the adjusted corrected discriminant of the quadratic form q*".

More generally, one deduces in essentially a trivial fashion from 16.5 the following statement.

Proposition 16.6. *We can and in a unique fashion associate with every prescheme S with a locally free module of finite rank E and with a quadratic form $Q \in \Gamma\, Sym^2(E)$ associate a section $\tilde{d}(Q)$ of $\det(E)^{\otimes 2}$ in such a way as to satisfy the following conditions:*

a) *compatibility with base change and functoriality with respect to isomorphisms of E.*
b) *If E is everywhere of even rank we have $\tilde{d}(Q) = d(Q)$ where $d(Q)$ denotes the discriminant of Q.*
c) *If E is everywhere of odd rank we have $2\tilde{d}(Q) = d(Q)$.*

N.B. It is not reasonable to announce the property of compatibility with base change without announcing at the same time, or even beforehand, the property of functoriality with respect to isomorphisms being given (étant donne) that the base change over E itself cannot be defined except module isomorphisms it will not be the same if we would *restrict* utselves to the case $E = O_S^n$ (which would not e convenient (suitable) for *the references*).

Definition 16.7. The section $\tilde{d}(Q)$ of $\det(E)^{\otimes 2}$ will be called the *corrected (adjusted) discriminant* of the quadratic form Q.

Corollary 16.8. *Let Q, Q' be two forms relative to E and E' suppose the parity of th ranks of E (resp. E') is constant over S. Then*

a) *if E and E' are not both of odd rank then we have $\tilde{d}(Q \otimes Q') = \tilde{d}(Q)\tilde{d}(Q')$*
" b)" *If E and E' are both of odd rank we have $\tilde{d}(Q \otimes Q') = 2\tilde{d}(Q)\tilde{d}(Q')$.*
The verification is trivial.

Now we can announce the principal result of the present section:

Theorem 16.9. *Let E be a sheaf of modules locally free of finite rank ovr prescheme S, $Q \in \Gamma\, Sym^2(E)$. The following conditions are equivalent:*

i) *Q is smooth.*
ii) *The modified discriminant $\tilde{d}(Q) \in \Gamma\ \det(E)^{\otimes 2}$ is invertible.*

iii) *If E is of constant rank n then there exists a surjective morphism $S' \to S$ such that the form $Q_{S'}$ deduced from Q by base change is isomorphic to the standard quadratic form in n-variables.*

iii bis) *as in iii) but saying (supposing) $S' \to S$ is faithfully flat of finite presentation (fppf).*

Corollary 16.10. *Let k be a field E a vector space of finite dimension over k, $Q \in Sym^2(E)$. The following conditions are equivalent:*

i) *Q is smooth.*

ii) *$\tilde{d}(Q) \neq 0$.*

iii) *there exists an extension k' of k such that $Q \otimes k'$ is isomorphic to standard form.*

iii bis) *as in iii) with k' a finite exension of k.*

This is a trivial consequence of 16.9. We note that if k is algebraically closed then iii) and [(iii) bis] can be replaced by a more interesting or intriguing condition: Q is isomorphic to the standard form.

Proof of 16.9. We may evidently suppose that E has constant rank n. We would obviously have (iii bis)⇒(iii)⇒(ii), taking into account that for the standard form the modified discriminant is one. We now prove that (ii)⇒(iii bis) also (i)⇔(ii).

Let $E_0 = O_S^n$, Q_0 a standard form over E_0, consider the functor $\text{Isom}((E_0, Q_0), (E, Q))$:
$\text{Sch}/S^0 \Rightarrow \text{Ens}$, with value at every S' over S formed from the set of isomorphisms $E_{0S'} \Rightarrow E_{S'}$ compatible with the forms $Q_{0S'}$ and$Q_{S'}$. It is immediate (without condition on E or Q that this functor is representable by a prescheme P affine and of finite presentation over S, which is a sub-prescheme of $W = W(\text{Hom}_{0S}(E_0, E))$ and a closed sub-prescheme of the open subset $\text{Isom}(E_0, E_1)$ of W.

The implication (ii)⇒(iii bis) will be proven if we should that if $\tilde{d}(Q)$ is invertible then P is faithfully flat over S: we take in fact $S' = P$.

Let us put to simplify $Q(E) = W(\text{Sym}^2(E))$ and let us define [de même] $Q(E_0)$ by the operation "transport of structure" we hae therefore a morphism $u \Rightarrow Q(u)$:

(∗) $\text{Isom}(E_0, E) \to \text{Isom}(Q(E_0), Q(E))$ and using the section q_0 of $Q(E_0)$ corresponding to Q_0 we find a morphism $u \Rightarrow Q(u)(q_0)$:

(∗) $\text{Isom}(E_0, E) \Rightarrow Q(E)$.

On the other hand the form Q corresponds to a section q of $Q(E)$ over S, and P is nothing else but the invrse image of $q(S)$ by (∗) as it follows trivially from definitions. (Which, besides, establishes in passing the announced representability of P as an affine prescheme and of finite presentation over S).

Let us now introduce the open subset

$$Q(E)^* = Q(E)_{\tilde{d}}$$

of $Q(E)$ corresponding to quadratic forms with invertible corrected discriminant representing the functor. ($S' \to$ set of sections of $\mathrm{Sym}^2(E'_S)$ with invertible corrected (adjusted) discriminant). Indeed by transport of structure we have $\mathrm{Isom}(E_0, E) \to \mathrm{Isom}(Q(E)_0^*, Q(E)^*)$ and since q is a section of $Q(E)^*$ (since the discriminant (adjusted) corrected of the form of type Q has value 1) the morphism $(*)$ can be factored indeed there is a morphism (into a morphism).

$(**)$ $\mathrm{Isom}(E_0, E) \to O(E)^*$ which is evidently of finite presentation since the two members (terms) are of finite presentation over S.

It now suffices to prove

Proposition 16.11. *With the previous notations (to be recalled) the morphism $(**)$ is faithfully flat.*

It will follow that if Q satisfies (ii), i.e. if q is a section of $Q(E)$, then P (deduced from $(**)$ by the base change q) is again faithfully flt over S which proves (ii)⇒(iii bis) and therefore also 16.9.

Proof of 16.11. Since the two terms are smooth over S it suffices to prove the flatness fiber by fiber which brings us to the case where S is the spectrum of an algebraically closed field k. We may evidently suppose that $E_0 = E$ so that $(**)$ takes the form $\mathrm{GL}(E) \to Q(E)^*$ these morphisms are deduced from natural operations of the group scheme $\mathrm{GL}(E)$ on $Q(E)^*$ by $u \to u(q_0)$ where q_0 is a section of $Q(E)^*$ corresponding to the standard form Q_0. But by the generic flatness theorem since $Q(E)^*$ is smooth over k, therefore reduced, there exists anopen and dense set u in $Q(E)^*$ over which the preceding morphisms are flat.

For every $u \in \mathrm{GL}(E)(k)$, i.e. for every automorphism u of E, $u(U)$ satisfies therefore the same condition and it suffices to move in order to establish the flatness of $(***)$ that the $u(U)$ cover $Q(E)^*$. But for this it suffices to prove that

$$\mathrm{GL}(E)(k) = \mathrm{Aut}(E)$$

acts transitively on $Q(E)^*(k)$, the set of quadratic forms over E^v with non-zero discriminant, i.e. that $(***)$ is surjective which also proves 16.11.

We are therefore reduced to proving

Lemma 16.11.1. *Let Q be a quadratic form inn-variables over an algebraically closed field such that $\tilde{d}(Q) \neq 0$ then Q is isomorphic to a standard form (i.e. [se met Fr] in standard form by a suitable choise of basis).*

If E is of even rank or if k is of characteristic $\neq 2$ then the hypothesis means that Q is non-degenerate and the conclusion can be found in Bourbaki.

In the opposite case (k of characteristic two and rank of E odd) we see that Q is degenerate, let E_1 be a straight line in E which is in the kernel of the associated form, E_2 a complement such that Q is in the form of a direct sum of the forms $Q_1 \oplus Q_2$, Q_1 of rank one and Q_2 of even rank. By 16.8 we see that Q_1 and Q_2 are of the corrected discriminant $\neq 0$, by what cam before it follows that Q_2 is isomorphic to standard form; on the other hand, Q_1 is non-zero so it is isomorphic to the standard form of degree one, thus Q is isomorphic to the standard form, which proves 16.11.1.

It remains to prove the equivalence of (i) and (ii) in 16.9. We may evidently suppose that$S$ is the spectrum of an algebraically closed field and we are reduced to proving in this case the:

Lemma 16.11.2. *Q smooth $\Rightarrow$ Q standard.*

By 16.2 we may assume that k has characteristic two and E is of *odd* rank. If Q is isomorphic to the standard form, we immeditely verify due to 16.3 tht it is smooth (the kernel of $E^v \Rightarrow E$ is of dimension one and Q is not identically zero ovr its kernel so that it has only the trivial zero).

On the other hand, still by 16.3, Q smooth implies that the restriction of Q to the kernel N of Q has only the trivial zero which evidently means that N is of dimension one and that $Q \mid N \neq 0$.

But by taking a complement of N we see immediately tht this implies that Q is isomorhic to the standard form. This proves 16.11.2 and achieves the proof of 16.9.

Nota Bene: We could have given 16.11.2 in the beginning as a corollary to 16.2 or 16.3 (which could be interchanged); then the part of this section (No.) independently of the "corrected discriminant" would be amalgamated at the beginning of the section.

Also you should know better than me to what extent the notion of the corrected discriminant at 16.9 are known, so as to give the correct credit. Perhaps there is another recent terminology?

I also remark that it is really the corrected (adjusted) discriminant that really deserves the name of discriminant, it is this one tht can be generalized to the discriminant of any form (see next or future [suivant Fr] section). The discriminant of a quadratic form (in the trminology here adopted, and which, if I am not mistaken (je ne me trompe [Fr]), is the recent terminology – I don't have Bourbaki at hand to check this point should be called the determinant of a quadratic form and not the discriminant. I would love to know your opinion about this question. If you agree, we use this occasion to correct at this point the recent terminology which induces (causes) an error (since until the last days Ihave without realizing it confused the discriminant and the determinant).

Lemma 16.12. *If we let the absolute group scheme $GL(n) = Aut(Z^n)$ act on $Q(Z^n)$ the stabilizer of the standard quadratic form is denoted by $O(n)$ and it is called the "absolute orthogonal group" (more generally for every quadratic form $Q \in \Gamma(Sym^2(E))$ we introduce the group subscheme formed by the $GL(E)$ stabilizer of Q denotes $O(Q)$ and called the orthogonal group scheme relative to the form O; if Q is isomorphic to the inverse imgae of the standard form in n-variables $O(Q)$ is isomorphic to $O(n)_S$). This being granted, it is immediate that two points Q, Q' of the first term in $(**)$ with value sinan S' have the same image in the second term $\Leftrightarrow$ (iff) we have $u' = uv$ with $V \in O(n)$ (S') taking into account 16.11 (and with the terminology that will be developed in more detail in EGA V and VI [illegible, ask AG] and awaiting this in SGAD IV we see then that the natural operations on the right of $O(n)_S$ on the first term of $(**)$ and the projection $(**)$ make $Isom(E_0, E)$ a principal homogeneous fibration ovr $Q(E)^*$ with group $O(n)^*_{Q(n)}$.)*

More specifically, GL(n) is a principal homogeneous fibration over $Q(n)^* = Q(Z^n)^*$ with structure group $O(n)^*_{Q(n)}$. It follows that (with the notations of the proof of 16.9) P is in fact a principal homogeneous fibration with the group $O(n)_S$, associated, in addition, canonically to the form Q (in a functorial fashion for the isomorphism of forms and in a manner compatile with base change. From purely formal agreements and from the "theory of flat descent" of EGA V (check reference with AG) we prove then that the functor $(E, Q) \Rightarrow P$ gives an *equivalence* of the fibered category of smooth quadratic forms on locally free modules of rank n with basis an arbitrary prescheme S with the fibered category of principal homogeneous fibrations over any prescheme S with group $O(n)_S$.

16.13. Let Q be a smooth quadratic form over an E locally free of finite rank, then we easily verify tht for $s \in X$, $O(Q)$ is smooth over an open neighborhood of s, except exactly in the case where $k(s)$ is of characteristic two and the rank of E at s is odd. Let us suppose that such cirumstances do not arise for any $s \in S$ then (and only then) the P anticipated above is smooth ovr S (being a homogeneous principal fibration ovr [under] for $O(n)_S$, thus smoothif the latter is such). Using "Hensel's lemma", it follows that we may then under conditions equivalent to 16.9 adjoin the equivalent condition (iii ter) as (iii) but supposing $S' \Rightarrow S$ *étale* and surjective. Indeed it is better (preserving always the previous hypothesis), since it follows from the general theory of reductive group schemes and principal homogeneous fiber bundles over them (cf. SGAD XXIV), that if Q Q is smooth then every point s has an open neighborhood U and an étale surjective *finite* morphism $S' \to U$ such that $S_{S'}$ has the standard form. If, for example, S is local, we can in (iii ter) assume in addition that $S' \to S$ is *finite*.

16.14. These results, that means those in 16.13 (beginning with condition (iii ter) for a smooth Q), become false [tombent en defaut] if we abandon the additional hypothesis of the rank and of the characteristic, for example if $S = \mathrm{Spec}(k)$, k an imperfect field of characteristic two, and that E is of rank 1 because the quadratic form at k^2 for $a \in k - k^2$ obviously cannot (???) be expressed in a standard form after a separable extension of k (note that $k(a^{1/2})$ is not separable over k). However, in the general case we can find a finite locally free and surjective morphism $S' \to S$ (indeed a principal homogeneous space (fibration) over the group scheme $\mu_{2s} \cong O(n)_S/SO(n)_S$ of square roots of unity over S. The base change $S' \to S$ has the effect of reducing the structure group of $O(n)$ to $SO(n)$ which is smooth) so that for $Q_{S'}$ the result of the local isotriviality mentioned before is true. In particular, if S is local then we can assume in (iii bis) that $S' \to S$ is *finite*.

Part III

Section 0

Invertible sheaves and divisors relative to projective and multiprojective fibrations linear systems of divisors

1) Invertible sheaves on a projective and multiprojective fibrations.
2) Representability of Div $\frac{L}{X/S}$: relative divisors on projective and multiprojective fibrations.
3) Linear systems of divisors and morphisms into projective fibrations.
4) Linear systems of divisors and invertible modules.

The results developed in this paragraph and in the following (ones) are already partly global in nature and they give, namely, complements desur (about) projective schemes using the global constructions of Chapter II and also have some result sof Chapter III and also purely local results of the present Chapter IV.

One of the aims of the present pragraph is to develop the language of "linear systems of divisors" connected on the one hand to the classificatin of morphisms into a projective fibration, on the other hand to the classification of invertiblel modules over a given prescheme. Let us note that the 'parameter schemes' really natural for the linear systems of divisors are the Brauer-Severi schemes which generalize projective fibrations and can be defined, for example, as fibrations that gecome isomorphic to a projective fibration after an étale surjective base extension.

Since their study uses descent theory developed in chapter V ([Tr] of the original design) and since also their classification is equivalent to the classification of group torsors of the projective group we postpone the study of such schemes and their connections with the notion of linear systems of divisors to Chapter VI of our scientific work. From a technical point of view, the main result of this section (paragraph) is Theorem 1.1, which determines the Picard group of projective fibration in terms of that of the base, and its first corollaries developed in Nos. 1 and 2.

Section 1

(Crossed out in the original until the point that we will indicate.)

Invertible sheaves and divisors on projective fibrations and [illegible] **projective** (tr: perhaps multiprojective) **linear systems of divisors.**

(1) Determination of invertible sheaves on projective fibrations. Application to the automorphisms of a projective fibration

Theorem 1.1. *Let S be a prescheme E a locally free module over S of finite rank ≥ 2 at every point $P = P(E)$ the projective fibration that it defines. Then for every invertible module L over P we may find a family $(S_n)n \in Z$ of open disjoint sets in S covering S indexed by Z and an invertible module M over S so thta the restriction of L to $f^{-1}(S_n)$ should be isomoprhic to that of $M \otimes O_P(1) = f^*(M)(n)$. Also, the family of the S_n is uniquely determined by these conditions, thus M (is determined) up to a unique isomorphism.*

Remark 1.2. If we make no assumption about the rank of E then S canonically decomposes into the sum prescheme of the opens S^0, S^1, S^2 such tht over them [???] sur ceux-ci the rank of E is respectively 0, 1 and ≥ 2. Then the determination of the invertible modules over P is reduced to the $f^{-1}(S^i)$ for $i = 0, 1, 2$. The case $i = 2$ is justifiable from 1.1, on the other hand $f^{-1}(S^1)$ is S^1 isomorphic to S^1 hence its Picard group is nothing else but $\mathrm{Pic}(S^1)$ finally $f^{-1}(S^0)$ is empty thus its Picard group is zero.

Corollary 1.3. *Under the assumptions of 1.1 let us assume that S is connected and non-empty. Then every invertible module L over P is isomorphic to a module of the form $f^*(M)(n)$ where $n \in Z$ and M is an invertible module over S. Also n is uniquely determined and M is determined up to a unique isomorphism by the giving of L.*

Another way to formulate this corollary is the following. Let us consider the natural homomorphisms $\mathrm{Pic}(S) \to \mathrm{Pic}(P)$ [illegible] $Z \to \mathrm{Pic}(P)$ the first one deduced from $f\colon P \to S$ the second one determined by the element $d(O_P(1))$ of $\mathrm{Pic}\, P$. We deduce a canonical homomorphism $\mathrm{Pic}(S) \times Z \to \mathrm{Pic}(P)$ defined anyway without any restrictive assumption on S or on E. This gives:

Corollary 1.4. *Under the conditions of 1.1 if $S \neq 0$ then the preceding homomorphism is injective and even bijective if S is connected.*

If we abandon the assumption about the rank being ≥ 2 it follows from 1.2 and 1.4 that the preceding homomorphism is again surjective if S is connected but not necessarily *injective, the kernel being isomorphic to Z, resp. to $\mathrm{Pic}(S)\times Z$ if E is of rank 1, respectively of rank zero.*

Let us prove 1.1 by starting from uniqueness. First of all if S is the spectrum of a field let us notice that $O(n)$ is not isomorphic to $O(m)$, i.e. $O(m-n)$ is not isomorphic to O_P que si [Fr] $n-m=0$. This comes from the fact that $O(1)$ is ample and dim $P \geq 1$ (from the assumption that rank $E \geq 2$): Indeed we may suppose $d = m-n \geq 0$ if we had $d>0$ then $O(d)$ would be ample and could not be isomorphic to O_P except if P is quasi-affine thus finite (since it is proper ovr K). This alreadyp roves the uniqueness of the family $(S_n)n \in Z$ considered in 1.1. For the uniqueness of M up to unique isomorphism, we are reduced to the case $S = S_n$, I say that in this case we have as isomorphism (uniquely determined in terms of the isomorphism $L \to f^*(M)(n)$ $(**)M \to f_*(L(-n)))$. Indeed the isomorhism $(**)L \to f^*(M)(n)$ defines an isomorphism $L(-n) \to f^*(M)$ and consequently an isomorphism of the second term in $(**)$ with $f_*(f^*(M))$ which by itself is isomorphic to M, (M being locally free) to $M \otimes f_*(O_P)$ since [or] [Fr] [ref] $f_*(O_P) \leftarrow O_S$, hence the isomorphism $(**)$.

Let us prove the existence of the (S_n), M. Due to the uniqueness already shown the question is local over S, i.e. we are reduced to proving the

Corollary 1.5. *Let E be locally free and of finite rank over S, $P = P(E)$, L an invertible module over P, $s \in S$, then there exists an open neighborhood U of s and an integer $n \in Z$ such that $L \mid f^{-1}(u)$ is isomorphic to $O_P(n) \mid f^{-1}(u)$.*

Of course since the rank of E at s is ≥ 2 the integer n is well defined.

In addition 1.5 is trivial since the rank of E at s is ≤ 1. Let us note that E, est dela frima [Fr], since the question being local we may suppose already$E = O_S^{r+1}$, hence $P = P_S^r$. By the brief procedure of Paragraph 8 [of EGA IV? Tr] we are reduced also to the case where S is noetherian. We proceed in two steps:

a) S is the spectrum of a field K. We see that L is [Nousavon que Fr] defined by a graded module of finite type L over the gradual ring $K[t_0, \ldots, t_r]$. We also see that the restriction L' of L to the affine space epointe [Fr] punctured (?) $E_K^{r+1} - \{0\}$ is nothing else but the inverse image of L by the canonical projection morphism $q\colon E_K^{r+1} - (0) \to P_K^r$ and is therefore an invertible module. Let $i\colon E^{r+1} - \{0\} \to E^{r+1}$ be the canonical immersion it follows from the fact that the affine ring $K[t_0, \ldots, t_r]$ of E^{r+1} is factorial, thus a fortiori its localization at the point 0 of E^{r+1} is factorial and from the fact that the latter ring is of dimension ≥ 2 that $i_*(L')$ is an invertible module thus corresponding to an invertible module [illegible letter] over $K[t_0, \ldots, t_r]$. In addition $M = \lceil(E^{r+1} - 0, q^1(L))$ is graded in a natural fashion, finally the homomorphism $L \to M$ is evidently an isomorphism at all the points of E^{r+1} distinct from zero.

Thus after replacing L by M we are reduced to the case of an invertible L. But $K[t_0, \ldots, t_r]$ being factorial, L is then free of rank one, initialy by ignoring (neglecting) its grading; but a standard lemma (which can be found in Bourbaki without a doubt) implies that it is given free of rank [illegible] as a graded module which implies that the associated module over P is isomorphic to an $O_P(n)$. From the editorial point of view it would be clumsy (maladroit [Fr]) to begin by considering an L of finite type.

Let us begin carvement by considering $L = \lceil(E^{r+1} - (0), q_*(L))$ defining the module $L = q_*(L))$ we see (Chapter II) that L is a graded module that determines precisely L if we prove that L is free of rank 1 as a graded module by the indicated reasoning.

b) General case. Is deduced from case a) due to III.4.6.5 by using the relation $H^1(P^r_K, O_P^{r_K}) = 0$ established in III.2. q.e.d.

A variant of 1.4. Let $Z(S)$ be the set of locally constant functions with integer values over S, we define an evident homomorphism: $Z(S) \to \mathrm{Pic}(P)$ the $n \in Z(S)$ correspond in fact to partitions $(S_n)n \in Z$ on disjoint open sets (among which some may be empty) and to such a partition one associates the invertible module $O_P(n)$ whose restrictions to $f^{-1}(S_n)$ is $O_P(n)$. We find thus a variant (* bis) $\mathrm{Pic}(S) \times Z(A) \to \mathrm{Pic}(P)$ and a statement more general and more satisfactory than 1.4 affirming that (under the conditions of 1.1) this is a bijection. (NB Since $S \neq 0$ then the canonical application $Z \to Z(S)$ associating to every $n \in Z$ the constant function of value n is injective, resp. surjective, and we recover 1.4 formally which should be expressed (or stated) in the most general form that I vien d'expliciter [Fr].

Let us remark also as a remark that in the language of the Picard scheme which will be introduced in Chapter V.1.1 in the preceding equivalent form s'enouce simplement [Fr] by saying that the canonical homomorphism $Z_S \to \mathrm{Pic}_{P/S}$ of constant group schemes Z over S into the Picard scheme, deduced from the section of the latter defined by $O_P(1)$ is an isomorhism.

Let P be a projective fibration over a field K. An invertible module L over P is said to be *of degree* n if L is isomorphic to $O_P(n)$; if $\dim P \geq 1$ this determines n in terms of L but if $\dim P \leq 0$ (i.e. P is empty or reduced to a point) then L is of degree n for every n. To say that L is of degree n means also, because of 1.1 and 1.2, that the class of L in $\mathrm{Pic}(P)$ is in the image of $\mathrm{Pic}\, S \times \{n\}$ for the homomorphism $(*)\mathrm{Pic}(S) \times Z \to \mathrm{Pic}(P)$ described above, i.e. that L is isomorphic to a module of the form $f^*(M)(n)$ where M is invertible over S. Also, if the fibrs of P are non-empty, i.e. E is everywhere of rank ≥ 1, then M is determined up to a unique isomorphism in terms of L as follows, again, from

1.1 and 1.2. (By the way A ce propos [Fr] I notice that it is proper to announce 1.1 also without any hypothesis about the rank of E: every invertible L over S can be taken in the form indicated in that statmenet; lousque [Fr] if the fibers of P are non-empty, i.e. the rank of E is ≥ 2 athen the partition of S is also determined uniquely by the choice of L. In this way th remark 1.2 is eliminated and passe [Fr] in the proof.)

Let $P^1 = P(E^1)$ be a second projective fibration, then the determination of $\mathrm{Pic}(P)$ allows in principle to determine the S-morphism $g\colon P \to P^1$, since these are defined by an invertible module $L(g^*O_P)$ over [illegible] and a homomorphism $f_*(L) \leftarrow E^1$ (such taht the associated homomorphism $f^*(E^1) \to L$ is surjective) modules an isomorphism of L. We say that $g\colon P \to P^1$ is of degree n if $L = g^*(O_P(1))$ is of degree n. It suffices evidently de savoir [Fr] determine the homomorphisms of degree n for every n given. Let us note tht if g is of degree n and P has fibers of dimension ≥ 1 we necessarily have $n \geq 0$ (since over a field K if dim $P \geq 1$ then $O_P(n)$ is generated by its sections only if $n \geq 0$.), of course we could restrict ourselves to the case where P has its fibers of dim ≥ 1 (by proceeding as in 1.2). Since we have $f_*(f^*(M)(n)) = M \otimes f_*(O_P(n)) = M \otimes \mathrm{Sym}^2(E)$ we see that the determination of the S-morphisms $g\colon P \to P^1$ is reduced to the determination of the couples (m, u) up to isomorphism where M is an invertible module over S and $u\colon E^1 \to M \otimes \mathrm{Sym}^n(E)$ is a homomoprhism such tat the corresponding homomorhism [illegible] $^*(E^1) \to f^*(M)(n)$ is an epimorphism. Then g determines a first invriant of a global nature over S, savoir [Fr] $(M) \in \mathrm{Pic}(S)$ and this invariant fixed by the chosen M, the corresponding g correspond to a certain subset of the quotient set $\mathrm{Hom}(E^1, M \otimes \mathrm{Sym}^n(E))/I(S, O_S)^*$, the passage to the quotient by the group $I(S, O_S)^*$ corresponds to "module isomorphism" in the description of the $g\colon P \to P^1$ via invertible modules (Nota bene: the endomorphisms, resp. automorphisms, of an invertible L over a projective fibration P correspond to sections, resp. invertible sections, of O_P over P or even to sections, resp. invertible sections, of $O_S \to f^*(O_P)$ over S.)

Special cases (particular cases)

1) $n = 0$ – then we must take the homomorphism $E^1 \to M$ that are surjective, i.e. everywhere non-zero modules isomorphism of M. We find exactly the morphisms $g\colon P \to P^1$ of the form h, where h is a section of P over S (savoir [Fr] those determined by the invertible quotient M of E^1). Thus the S-morphisms of degree 0 of P into P^1 are the constant morphisms relative to S.

[Here ends the crossed out part Translator]

2) $n = 1$ – we must take the homomorphisms $E^1 \to E \otimes M$ or $L \otimes M$ tht are *surjective* as one erifies immediately and the corresponding homomorphism $g\colon P \to P^1$ is nothing else but the composition $P(E) \to P(E \otimes M) \to P(E^1)$ where the first homomorphism is the canonical isomorphism described in Chapter II and the second is the canonical closed immersion deduced from the epimorphism $E^1 \to E \otimes M$. If we call linear the homomorphisms from P into P^1 which can be so described as such a composition we see that the morphisms $g\colon P \to P^1$ which are of degree 1 are exactly the *linear* ones.

To finish let us determine the isomorphisms of P with P^1.

Theorem 1.6. *Let S be a prescheme, $P = P(E)$ and $P^1 = P(E^1)$ two projective fibers over S defined by E and E^1 locally free of finite type. Then every S-isomorhism g [illegible] $: P \to P^1$ is definable as a composition $P(E) \to P(E \otimes M) \to P(E^1)$ where M is an invertible module over S, the first homomorphism is the canonical isomoprphism of Chapter II and the second is the isomorphism deduced from an isomorphism $u\colon E^1 \to E \otimes M$. Since the fibers of P are non-empty (resp.of* $\dim \geq 1$*) therefore M (respectively the couple (M, U)) is determined up to a unique isomorphism in terms of g.*

According to the above considerations we are reduced to proving that g is of degree one which reduces us to the cse where S is the spectrum of a field and also (bien sur [Fr]) we may suppose that dim $P \geq 1$. But let us note that $O_P(1)$ is then intrinsically characterized (i.e. independently from the way that P is reduced as a projective fibration) as the generator of Pic(P) (between the two generators $O_P(1)$ and $O_P(-1)$) which is ample; consequently if $g\colon P \to P^1$ is an isomorphism then $g^*(O^1_P(1))$ is isomorphic to $O_P(1)$ and we (gagne) [Fr] – In local form less savante [Fr] we may announce:

Corollary 1.7. *Under the conditions of 1.6 every S-morphism $g\colon P \to P$ can be described in a neighborhood of each $s \in S$ usign an isomorphism $u\colon E \mid U \simeq E^1 \mid U$ the latter being well defined module multiplication by an element of $\lceil(U, O_U)^*$. In particular:*

Corollary 1.8. *Let S be a prescheme, $P = P(E)$ a projective fibration over S defined by E locally free of finite type, u an automorphism of P. Then u is determined in a neighborhood of every point $s \in S$ by an automorphism u of $E \mid U$ the latter being well defined by u module multiplication by an element of $\lceil(U, O_U)^*$.*

Remark 1.9. In 1.8 we could easily deduce that the group functor $\mathrm{Aut}_S(P)$ over S is representable by an affine prescheme of finite presentation over S, which can be also interpreted as the quotient group scheme of the linear group scheme $G \mid (E)$ by its center G_M. The group prescheme is called the prescheme of projective groups or simply projective group defined by E and is denoted $GP(E)$. If E is free $E \cong O_S^{r+1}$, thus $GP(E)$ is nothing else but the group prescheme $GP(r)_S$ deduced by base change $S \to \mathrm{Spec}\, Z$ of the analogous group scheme $GP(r)$ over $\mathrm{Spec}\, Z$ called the absolute projective group.

End of Appendix 1. Marginal remark next to Remark 1.9 partly illegible [illegible] $P(E^v \otimes E)$ defined by the non-vanishing of the "determinant".

Section 2
Relative divisors and invertible shaves on projective and multiprojective fibrations

2.1. Let, as in No. 1, $P = P(E)$ E locally free over S of rank ≥ 2 everywhere. We propose to determine the set $\mathrm{Div}(P/S)$ of relative divisors ≥ 0 over P p.r. (pr rapport ? with respect) to S. We see that to give such a divisor is the same as to give an invertible module L over P with a section ϕ of L transversally regular. But according to 1.1 (ignoring a possible partition of S if S is not connected) L is isomorphic to a $M \otimes O_P(n)$ where$M$ is an invertible module over S in addition determined up to a unique isomorphism in terms of L. In addition we see tht we have also the canonical isomorphisms

$$(*) \qquad f_*(L) \simeq M \otimes f_*(O_P(n1)) \simeq M \otimes \mathrm{Sym}^n(E)$$

so that to give a section ϕ of L is the same as to give a section ψ of $m \otimes \mathrm{Sym}^n(E)$. Taking into account the fact that the fibrs of P/S are integral we see in addition tht ϕ is transverse regular (i.e. regular on each fiber) if and only if $\psi(s) \neq 0$ for every $s \in S$ or which is the same if and only if the homomorphism $\psi\colon M^v \to \mathrm{Sym}^n(E)$ deferred by ψ is "universally injective", i.e. locally an isomorhism onto a direct factor, or what is the same if its transpose $\psi\colon \mathrm{Sym}^n(E)^v \to M$ is surjective.

We say that a relative divisor D over P is *of degree* n if $O_x(D) = L$ is of degree n in the sense of the previous No. Since $D \geq 0$ this implies $n \geq 0$ since [illegible] (on avait ???) if we had (?) $n < 0$ every section of L over X were zero. By 1.1 if D is a relative divisor ≥ 0 over P then there exists a unique decomposition of D into the disjoint sum of open subsets $S_n (n \in N)$ such that for every $n \in N$, $L/P/S_n$ is fo degree n. This reduces the determination of the set of relative divisors ≥ 0 to the case of relative divisors ≥ 0 of given degree n.

This text replaces of course the 'abstraction faite' above (translated *ignoring*).

This being granted the above reflections give the

Proposition 2.2. *Under the above hypotheses we have a one-to-one correspondence between the set $\mathrm{Div}^n(P/S)$ of relative divisors ≥ 0 of degree n over P and of the set of invertible quotient modules M of $\mathrm{Sym}^n(E)^v$ (or, which is the same, of invertible submodules locally direct factors M^v of $\mathrm{Sym}^n(E)$.*

If D and M correspond to each other then $O_P(D)$ is canonically isomorhic to $M \otimes O_P(n)$ and the section S_D is identified by this isomorhism a ce q'on devin [Fr] taking into account $(*)$.

Let us notice that this description is compatible with leut [illegible] base change in S. Taking into account the interpretation of invertible quotient modules of $\mathrm{Sym}^n(E)^v$ as sections over S of $P(\mathrm{Sym}^n(E))$, we find here: (oui Fr)

Corollary 2.3. *The subfunctor $\mathrm{Div}^n_{P/S}$ of $\mathrm{Div}^+_{P/S}$ is representable canonically by $P(\mathrm{Sym}^n(E)^v)$. Taking into account the considerations of 2.1, it follows that:*

Corollary 2.5. *$\mathrm{Div}^+_{P/S}$ is representable canonically by the S-prescheme sum of the $P(\mathrm{Sym}^n_{P/S}(E)^v)$, $n \in N$.*

Corollary 2.5. *Let us now suppose that we are given a finite family $(E_i)_{i\in I}$ of locally free modules over S, hence (d'ou'des Fr) $P_i = P(E_i)$ and a $P =$ fibered product of the P_i also the multiprojective fibration over S defined by the (E_i). For simplification of notations we denote by $O_i(n)$ the inverse image sur (Fr) P of the invertible module $O_{P_i}(n)$ over P_i. For every system of integers $n = (Ni)_{i\in I} \in Z^I$ we put $O_P(n) = \otimes_i O_i(n_i) = \otimes_{I_{O_S}} O_{P_i}(n_i)$ ceci pose [Fr], 1.1 generalizes as follows:*

Proposition 2.6. *Let us assume that the E_i have (partout Fr) rank ≥ 2. Then for every invertible module L over the multiprojective fibration P there exists a decomposition of S into the disjoint sum of open sets S_n, $n \in Z^I$ and an invertible module M over S such that $L/P/S_n$ is isomorhic to $O_P(n)/P/S_n$. Also the S_n are determined uniquely and M is determined up to a unique isomorphism.*

The proof consists in an immediate reduction to 1.1. Under the conditions of 2.6, we may therefore associate to every invertible L over P a 'multidegree' $n = (Ni)_{i\in I} \in Z(S)^I$ which characterizes L up to a unique isomorhism provided $\mathrm{Pic}(S) = 0$. also we may interpret the Ni (called the "partial degree of L with respect to the factor P_i of index i") if we take for each i a section g_i of P_i over S (NB such sections exist in any case locally over S) and if we note that this system defines for each i an S-morphism $g_i\colon P_i \to P$; this granted (cui pose F4) we have $N_i = \mathrm{dig}\, h_i^*(L)$ we point out that in general the n_i are not integers but they are locally constant functions of S into Z.

Proceeding as in No. 1, we may deduce from 2.6 the determination of a morphism of one multiprojective fibration into another and in particular the determination of the automorphisms of multiprojective fibrations. More interesting for us because of par 25 about the resultant (?) and discriminant of forms will be the determination of relative divisors ≥ 0 on a multiprojective fibration.

Corollary 2.8. *If D is a relative divisor over P we define its multidegree as that of $O_P(D)$. As above, the determination of $\mathrm{Div}^+(P/S)$ is reduced to that of $\mathrm{Div}^n(P/S)$ for a given multidegree $n \in I$ which gives an isomorphism $L = O_x(D) \approx M \otimes_{O_S} O_P(n)$.*

But we have due to Chapter II, Par. 2 (∗∗) $f_*(D) = M \otimes f_*(O_P(n)) = M \otimes_i Sym_i^n(E_i)$. Proceeding now as in 2.2, we find the

Propositioin 2.9. *With the preceding notations, to be recalled, we have one-to-one canonical correspondence between the set $Div^n(P/S)$ of relative divisors of multidegree n over P and the set of invertible quotient modules M of $\otimes_i Sym_i^n(E_i)^v$ (or, what is the same. . .) If D and M correspond to each other, then $O_P(D)$ is canonically isomorphic to $M \otimes O_P(n)$ and s_D is identified then a'ce qu'on devine [Fr], taking into account (∗∗).*

Corollary 2.10. *The subfunctor $Div^n_{P/S}$ of $Div^+_{P/S}$ corresponding to relative divisors of multidegree n is canonically representable by the projective fibration $P(\otimes_i Sym_i^n(E_i)^V)$ and $Div^+_{P/S}$ is canonically representable by the sum prescheme of the latter for $n \in N^I$.*

Corollary 2.11. *The preceding very simple determination of $Div_{P/S}$ is due to the very simple structure of Pic($P?S$) (indeed to the "discrete" structure of the Picard prescheme $Pic_{P/S}$ . . .) We may, abstracting from the reasoning just done, (abstraive le raisonment fait) [making the reasoning done abstract – Grothendieck's art of making things as general as possible – translator's remark] wich reduces essentially to establishing a relative representability (with respect to Pic).*

To do this let us take a morphism $f: X \to S$ proper and flat of finite presentation *and* an invertible module L over X. We propose to determine the subgroups $\mathrm{Div}^L(X/S)$ of $\mathrm{Div}^+(X/S)$ formed by relative positive divisors such that $O_X(D)$ is isomorphic to a module of the form $L \otimes_{O_X} M$ where M is an invertible module over S (depending on D). We assume that we have $f_*(X_X) \leftarrow O_S$ which implies that the above M (ci-dessus M [Fr]) is determined up to a unique isomorphism by D since $M = f_*(L^{-1} \otimes O_X(D))$. To give D corresponding to an L given is thus reduced to giving a transversally regular section ϕ of $L \otimes M$. But we have, because of Chapter III, Par. 7, that there exists a module Q of finite presentation over S whose formation commutes in addition with every base change and an isomorphism of functors in G (a quasi-coherent module variable over S) $f_*(L \otimes G)$ – $\mathrm{Hom}(Q, G)$ (To tell the truth in the loc. cit. we suppose that S is locally neotherian but if we get rid of this hypothesis in an evident way by a brief procedure taking into account the commuting of Q with base change). In particular, to give ϕ is equivalent to giving a homomorphism $\Psi: Q \to M$. A necessary condition for ϕ to be transversally regular and one which is sufficient if the fibers of X are integral is that ϕ should be $\neq \phi$ fiber by iber, which in terms of Ψ means simply that Ψ is surjective thus that Ψ corresponds to a section of the projective fibration $P(Q)$ over S. We obtain therefore the

Proposition 2.12. *In the above notations $Div^L(X/S)$ is in a canonical bijective corre-*

spondence with the set of sections of $P(Q)$ over S corresponding to a quotient module M of Q such that the section ϕ of $L \otimes M$ defined by $\Psi: Q \to M$ should be transversally regular. Let us suppose now that the hypothesis $O_S \to f_*(O_X)$ is encove [Fr] even (???) true after every base change or what reduces to the same by III.7 that we have the condition $k(s) \to H^0(X_s, O_{X_s})$ for every $s \in S$. Thus 2.12 applies equall well to every X_s^1/X^1 by an arbitrary base change $S^1 \to S$. We obtain therefore the

Theorem 2.13. *Let $f: X \to S$ be a morphism of finite presentation proper and that satisfying $(***)$ above L an invertible module over X, let us consider the subfunctor of $\mathrm{Div}^L_{X/S}$ of $\mathrm{Div}^+_{X/S}$ defined above in terms of L. There exists a module of finite presentation Q over S such that the preceding functor is representable by a sub-prescheme open retrocompact of the projective fibration $P(Q)$; since the fibers of X/S are geometrically integral thus $\mathrm{Div}^L_{S/S}$ is representable by the same fibration (by the fibration itself lui même [Fr]).*

The last assertion results immediately from the one that we have given before. For the general case we have already noted that we have a monomorphism $\mathrm{Div}^L_{S/S} \to P(Q)$ and we are reduced to proving that the latter is representable by an open quasi compact immersion, which reduces us to proving that if we take a section of $P(Q)$ over S, i.e. an invertible quotient module M of Q, d'ou [Fr] a section ϕ of $L \otimes M$ non-vanishing at any fiber then the subfunctor [illegible] of the final functor S over $(\mathrm{Sch})/S$ "consisting in making ϕ transversally regular" is representable by a retrocompact open subset of S. But the fact that it is representable by an open set gives the fact that f is proper and that the transverse regularity is an open condition (cf. par. 11...) the retrocompactness is seen immediately by reduction to the noetherian case.

Section 3
Linear systems of divisors and morphisms into projective fibrations

Let D be a family of divisors over X/S parametrized by T (we imply *in what follows positive*). A point $x \in X$ is called "fixed point" for this family of divisors if $\mathrm{pr}_1^{-1}(x) \subset D$ essentially so hat the set of non-fixed points is a complement of $\mathrm{pr}_1(X \underset{S}{\times} T - D)$, consequently if $T \to S$ is universally open (for example flat locally of finite presentation) then the set of fixed points is closed. We say that the family of divisors is without fixed points if the set Z of fixed points is empty. If Z is closed then $X - Z$ is the biggest oipen set such that the family of divisors of $X - Z$ parametrized by T induced by the given family in an evident sense is without fixed points. If the family D is "without fixed points" and if T is flatand locally of finite presentation over S with fibers (S_1) and geometrically irreducible then D is also a divisor relative to X (for $\mathrm{pr}_1 \colon X \underset{S}{\times} T \to X$): indeed D is defined locally at a point $z \in \operatorname{supp} D$ by one equation $\phi = 0$ and the equation induced on the fiber $T_{s\,k(x)}$ of the point $x \in X$ (over $s \in S$) is non-nilpotent at z (since otherwise $\mathrm{pr}_1^{-1}(x)D = V(\phi_x)$ would contain set-theoretically a neighborhood of Z in $T_{k(x)}$ therefore would contain $T_{s\,k(x)}$ i.e. x would be a fixed point which it is not) but since $T_{s\,k(x)}$ is irreducible and (S_1) it follows that ϕ_x is $O_{T\,k(x)}$-regular at Z. We obtain therefore a family of divisors of T/S parametered by X, i.e. a morphism

$$X \to \mathrm{Div}_{T/S}.$$

In the general case where the family of divisors of X may have fixed points, we obtain a family of divisors T/S parametered by $X - Z$, i.e. a morhism $x - Z \to \mathrm{Div}_{T/S}$ by replacing in the previous definition X by $X - T$.[74] Anyway the above proof shows that $X - Z$ is exactly the greatest open set U of X such that $d \mid U \underset{S}{\times} T$ is a divisor relative to U, i.e. such that its symmetric (image) ${}^tD$ is a family of divisors of T/S parametrized by U/S. We may remark that if X and T are both flat locally of finite presentation over S with fibers (S_1) and geometrically irreducible the symmetry $D \to {}^tD$ gives a one-to-one correspondence between families of divisors of X/S parametrized by T which are without fixed points and families of divisors of T/X parametrized by X which are without fixed points. If in this statement we wish to get rid of the specific assumption made about the fibers of X/S and T/S, it is convenient to replace the "fixed points" by "fixed points in a more general or extended sense" by understanding that by a fixed point in a general sense

[74] I think $X - Z$ [Tr].

of D an $x \in X$ such that D is not a relative divisor to X at all the points of $\mathrm{pr}_1^{-1}(x)$. IF W is open, the points of $X \underset{S}{\times} T$ where D is a relative divisor with respect to X then the set of fixed points in the extended sense of D is equal to $\mathrm{pr}_1(x \times_S T - W)$ since $T \to S$ is proper it is therefore a closed subset Z' of X. In every case (any case) we obtain a familyof divisors of T/S parametrizeed by $X - Z'$. The assumption that the fibers of T/S are (S_1) and geometrically irreducible serves precisely to insure tht $Z = Z'$ (fixed points = fixed points in extended sense). Geometrically, let us suppose for simplicity that $S = \operatorname{Spec} k$, k an algebraically closed field which is allowed for T/S flat and of finite presentation, by a base change, to say that $X \in X(k)$ is a fixed point (resp. fixed point in an extended sense) means that $x \in \operatorname{supp} D_t$ for every $t \in T(k)$ (respectively, that there exists a prime cycle T' associated to T such that $x \in D_t$ for every $t \in T'(k)$). An omission: The formation of the set of fixed points Z is compatible with base change in S; on the other hand $X - Z$ (assumed open, e.g. T/S flat locally of finite presentation) is universally schematically dense in X relative to S. This last fact results from Par. 11[75] andfrom the fact that for every $s \in S$ Z_s does not contain any point of X_s associated to O_{X_S} (indeed the support of divisor over X_s does not contain any such point). In the case where T is a projective fibration $Q = P(F)$ $D_{Q/S}$ is representable by the sum of $P(\mathrm{Sym}^n(F^v)) = P(n)$, we find a morphism $X - Z \to P(n)$. We say that the family of divisors D is *of degree* n if the preceding morphism factors by $P(n)$; if $X \not= \phi$ hence $X - Z \neq \phi$ the n in question is well defined by D. Besides, in addition, to define this notion we do not strictly need for rigor the result of representability announced above but only ot have defined the canonical monomorphisms

$$p(n) \to \mathrm{Div}_{p^v/S}$$

(N.B. we put$P = P(F^v)$ hence $Q = P(F) = P^v$ with the notations of the previous Nos.) We call *a linear system of divisors over X/S parametrized by the projective fibration $O = P^v$ every family of divisors over X/S parametrized by O which is of degree one, i.e. defining $f\colon X - Z \to P$.* Therefore to such a linear system of divisors and if the fibers of P^v are $\neq \phi$[76] is associated to a rational map of X into a projective fibration. Indeed, even better as a rational map [illegible] "pseudo-morphisme rel: S". By the very construction $D \mid (X - Z) \underset{S}{\times} P^v$ is nothing else but the inverse image by $(f \mathrm{x} \mathrm{id}_{P^v})$ of the canonical divisor (the incidence divisor) H over $P \underset{S}{\times} P^v$. Hence the knowledge of $f\colon X - Z \to P$ allows us to reconstruct at least the family of divisors of $X - Z$ induced by D so that if the family is without fixed points it is determined by the associated morphism $f\colon X \to P$. Let us note that we obtain evidently a one-to-one correspondence between linear systems of divisors

[75] Tr: make reference more precise
[76] Illegible

X/S parametrized by P^v and morphisms $f: X \to P$ such taht if $(f \underset{S}{\times} \mathrm{id})^{-1}(H)$ is a relative divisor over $X \underset{S}{\times} P^v$ with respect to P^v. This condition can be verified fiber by fiber and we obtain:

Proposition 3.1. *We have a one-to-one correspondence between the linear systems without fixed points of divisors over X/S parametrized by P^v and the morp;hisms $f: X \to P$ having the following property: for everys $\in S$, denoting by k an algebraic closure of $k(s)$ and for every associated prime cycle X' of X_k, $f(X') \subset P_{\bar{k}}$ is not contained in any hyperplane of $P_{\bar{k}}$. (If X has geometrically integral fibers this can be stated simply by saying that $f(X_{\bar{k}})$ is not contained in any hyperplane of $P_{\bar{k}}$).*

In general (i.e. if $Z \not\phi$) we can no longer affirm that the knowledge of f determines the family of divisors. The most trivial case of that where$P^v = S$ is of relative dimension zero. To give a linear system of divisors of X/S parametrized by S is equivalent to giving a relative Cartier divisor D over X relative to S, the associated morphism is the projection $X - D \to S$ and we see that the knowledge of this morphism (which includes knowing its domain of definition) does not determine D. In this case also let us suppose for simplicity that X is reduced the domain of definition of f considered as a rational map of X into $P = S$ is not $X - D$ but X. In order to eliminate this type of unpleasant phenomena we limit ourselves to linear systems of divisors "*without fixed components*". In general if $S = \mathrm{Spec}(k)$ to give a family (not necessarily linear) of divisors of $X \mid S$ parametrized by T we call a "fixed component" of the family every irreducible component of codimension one of the set Z of fixed points of the family; we say that the family is "without fixed component" if it is without fixed component, i.e. if $\mathrm{codim}(Z, X) \geq 2$. this terminology can be extended immeditely to the case where S is arbitrary by considering fiber by fiber. The property of being without a fixed component is evidently stable under base change.

Proposition. *Let us suppose $X \to S$ is flat locally of finite presentation with fibers (S_2) and let D be a linear system of divisors without fixed components over X/S parametrized by P^v. Then D is uniquely determined by the knowledge of the corresponding morphism $f: X - Z \to P$ (Z = set of fixed points) and even by the knowledge of the class of f as a "pseudo-morphism relative to S", $X - Z$ is the domain of definition of the said class.*

For this notation and the *sorite* of "pseudo-morphism relative to S" see section [20.10] of EGA IV.[77] We must prove that if D' is another linear system of divisors without fixed component parametrized by P^v defining $f: X - Z' \to P$ and if f and f' coincide on an open set $U(X - Z) \cap (X - Z')$ scheme–theoretically dense relative to S then $D = D'$.

[77]Only 20.1–20.6 exists in EGA IV (Tr)

Indeed since P is separated over S we may take $U = (X - Z) \cap (X - Z') = Z - Z''$ where $Z'' = Z \cup Z'$. Since Z'' is of codimension ≥ 2 over each fiber and since X has (S_2) fibers, it follows that for every $x \in Z''$ the fiberX_S is of depth ≥ 2 at x. We may certainly conclude (using the fact that X is flat locally of finite presentation over S) that every divisor over X (not necessarily transversal to the fibers) is known once we know its restrictions to $X - Z$, which gives the wanted conclusion.

Let J be the ideal which defines D, it evidently suffices to show that $J \to i_*(J \mid X - Z'')$ is an isomorphism (where $i: X - Z'' \to X$ denotes the canonical immersion), now the homomorphism $J \to \vartheta_x$ can be reconstructed in effect by applying the functor i_8 to $J \mid X - Z'' \to \vartheta_x \mid x - Z''$. But since J is invertible, it is flat over S and $X \in Z'' \Rightarrow \mathrm{prof}_x J_s \geq 2$. It is enough, therefore, to prove the:

Lemma 3.1. *Let $X \to S$ be of finite presentation, let F be a module over finite presentation over X, flat relative to S, T a closed subset of X.*

Let us assume that for every $x \in X$ over $x \in S$ we have $\mathrm{prof}\, xF_s \geq 1$ (resp. $\mathrm{prof}\, xF_s \geq 2$). The canonical homomorphism $F \to i_*(F \mid X - T)$ is injective (resp. bijective), where $i: X - T \to T$ ($i: X - T \to X$ should be Tr) is the canonical immersion.

bf Proof of the lemma: We may suppose S, X to be affine and by a brief procedure we suppose that S is noetherian.

Then the hypothesis implies by Par. 6 that we have $\mathrm{prof}\, xF \geq \mathrm{prof}\, xF_s$ for every $x \in X$ over $s \in S$, thus $\mathrm{prof}\, xF_s \geq 1$ (resp. ≥ 2) if $x \in T$. We conclude therefore by paragraph 5 of EGA 5. (NB: Pour bien faire this lemma ought to be in paragraph 11 under the heading: elimination of noetherian hypothesis. . .) (EGA IV see e.g. 11.3 [Tr]). It finally remains to verify the last assertion of Prop. 2 [illegible] that $X - Z$ is exactly the domain of definition of the rational map relative to S defined by f. Let $U \supset X - Z$ be its domain, it follows therefore from Proposition 1 that $U \to p$ is associated to a linear system of divisors D' over U/S parametrized by p^v and we have $D' \mid (X - Z)x_s p^v = d \mid (X - Z)x_s p^v$. Applying the uniqueness result (already proven) to D' and $D \mid U x_s p^v$, we see that the two latter divisors are equal, thus $D \mid U x_s p^v$ does not have fixed points, i.e. $Z \cap U = \phi$ thus $U = X - Z$. q.e.d.

I regret (I repent) to have given the proposition in a messed up ([Tr]: the original is in much more picturesque off-color French.) form half way between the classical hypothesis and natural hypothesis and without giving the converse this I propose to announce:

Proposition 3.3. *Let $X \to S$ be flat locally of finite presentation $Q = P^v = P(E^v)$ a projective fibration over S defined by a locally free module of finite type $F = E^v$. Let us consider the set ϕ of linear module of finite type $F = E^v$. Let us consider the set ϕ*

of linear systems D of divisors over X parametrized by Q such that the set Z of fixed points of D satisfies the property: $z \in Z$ implies prof $zO_{X_s} \geq 2$ (where s is the image of z in S). Let $\not{U}$ be the set of pseudo-morphisms f relative to S of X into P such that the domain of definition $U = \dots f$ satisfies the condition $z \in X - U \Rightarrow z\ (\text{prof} O_{X_s}) \geq 2$ and $f_u = f \mid u\colon u \to p$ satisfies the non-degeneracy included in 3. Let us consider the natural map $D \to f_D$ of ϕ into U then:

a) *This map is injctive and for $D \in \phi$ the set of fixed points Z is nothign else but the complement U of the domain of definition of f_D.*
b) *Let $f \in \not{U}$ and let U be the open set over which f is defined and such that $z \in X - U$ implies prof $O_{X_s}, z \geq 2$ for example $U = U(f)$, the domain of definition of f). In order that f should give a $D \in \phi$ it is necessary and sufficient that putting $L_U = f_U(O_P(1))$ (where $f_U\colon U \to P$ is the morphism induced by f) the module $i_*(L_U)$ over X is invertible (where $i\colon U \to X$ is the canonical immersion). We remark that if the fibers of X over S satisfy (S_2) for example if they are normal, see geometrically [voire Fr] normal, the depth condition considered over a closed set Z of X in the proposition simply means that for every $s \in S$, Z_s is of codimension ≥ 2 in X_s; ϕ is therefore the system of linear systems of divisors for P which are without fixed components. On the other hand, if $S = Spec(k)$ and if X is normal then for every rational map $f\colon X \to P$ the set of definition $U(f)$ satisfies codim$(X - U(f), X) \geq 2$ (II.7) so that in this case U is formed by the set of all the rational maps of X into P.*

The proof of a) has already been given. In order to prove b) let us note that the formation of $i_*(L_U)$ commutes with every flat extension S' of S(ref) at least if $u \to x$ is quasi-compact the case to which we reduce without difficulty so that the condition to consider is invariant under base change faithfully flat quasi-compact (qu cp). We take $S' = P^v$ and we note that the hypothesis that $i'_*(L'_U)$ is invertible does not change if we replace L'_U by $L'_U \underset{S'}{\text{x}} M'$ where M' is an invertible module over S' so that

$$i_* \left(L'_U \underset{S'}{\text{x}} M'\right) \simeq i'_* \left(L'_U \underset{S'}{\text{x}} M'\right)$$

We take $M' = O^v_P$ so that the mentioned condition means also that $i'_*(N')$ is an invertible module where

$$N' = \left(f_u \underset{S}{\text{x}} \text{id}_P\right)^* (O_P \times P^v(1,1)).$$

But $0(1,1)$ is precisely the invertible module defined by the *canonical divisor* [(word usage PB)] H of $P^v \underset{S}{\text{x}} P$ such that N is nothing else but the invertible module defined by $D' = (f_u \times \text{id}_P)(H)$. If f gives $D \in \phi$, D' is nothing else but $D \mid U \underset{S}{\text{x}} P^v$ therefore $N' = N \mid U \times P^v$

where N is the invertible module over $X \underset{S}{\times} P$ defined by D and it follows form Lemma 3.2 above applied to $P \underset{S}{\times} P^v \to P$ that we have $i_*(N') \simeq N$ therefore $i'_*(N')$ is invertible. Conversely, if this condition is satisfied we prove that f gives a $D \in \phi$ or what evidently reduces to the same thing that the divisor D' can be extended to a relative divisor with respect to P^v over $X \underset{S}{\times} P$. It reduces to the same to say that it extends to a divisor D over $X \underset{S}{\times} P$ since D will automatically be a relative divisor with base P^v as results from the fact that U contains elements associated to O_{X_S}, $s \in S$, a condition that is stable under base change and in particular by $s' = P^v \to S$. But it follows fimmediately from Lemma 4.2 above that D' extends to a divisor D if and only if D' extends to an invertible module ou encore $i'_*(N')$ is invertible. It would be necessary to extend to a divisor D if and only if D' extends to an invertible module ou encore $i'_*(N')$ is invertible. It would be necessary to edit the end of the proof in terms of necessary and sufficient condition (without referring to it twice as I did) and first of all release the:

Corollary 3.4 of Lemma 3. *2. Let us suppose that $g: X \to S$ is flat and locally of finite presentation. Let T be a closed subset of X such that $x \in T$ implies proof $O_{X_s} \geq 2$ (where $s = g(x)$) let $U = X - T$ and let $i: U \to X$ be the canonical immersion. For every locally free module of finite type L over X let us consider its restriction $L' = L \mid U$. Then*

- a) *the functor $L \to L'$ is fully faithful and for everyL the canonical homomorphism $L \to i_*(L')$ is an isomorphism. In order that L should be of rank n it is necessary and sufficient that L' should be such.*
- b) *Let L' be a locally free module over X then L' is isomorphic to a restriction of a locally free module L if and only if $i_*(L')$ is locally free.*
- c) *Let us suppose that L' is an invertible module associated to a divisor D' over U. Then the condition mentioned in b) is also necessary and sufficient in order that D' should be a restriction of a divisor D over X which will therefore be unique (and is equal to the scheme theoretic closure of D' in X). For D' to be a divisor relative with respect to S it is necessary and sufficient that D should be such. We simply use the fact that every L satisfies the announced hypothesis for F in Lemma 3.2.*

Corollary 3.5. *Let us assume that the local rings of X are factorial (for example X regular). Thent he map $\phi \to \not{U}$ is bijective. In particular if X is a regular prescheme locally of finite type over a field k and P is a projective fibration over k there is a one-to-one correspondence between the set ϕ of linear systems of divisors with no fixed components over X parametrized by P^v and the set U of rational maps of X into P that over k do not factor thorugh any hyperplane of P_k. Indeed since the local rings of X are factorial it follows that every invertible module over U extends to an invertible module over X*

so that the condition mentioned in b) is automatically satisfied. On the other hand, by Auslander-Buchsbaum a regular local ring is factorial.

Section 4
Linear Systems of Divisors and Invertible Modules

Using the results of (Section 1) No. 1, we shall give a complete description of linear systems over X in terms of invertible sheaves over X. We may evidently suppose that $P^v \to S$ is surjective, then $X \underset{S}{\times} P^v \to P^v$ is also such. It is also (Fr 144) and according to the generalities of 20.3 (Reference hard to locate, ask AG for help)[78] *to give a divisor D over $X \underset{S}{\times} P^v$ reduces to giving an invertible module N over $X \underset{S}{\times} P^v$ and a regular section ϕ of the latter. The assumption that D is a linear system of divisors over X parametrized by P^v can be expressed therefore by the two conditions*

1) *the $\phi_t (t \in P^v)$ induced by ϕ on the fibers of $X \underset{S}{\times} P^v$ over P^v are regular (which entails that ϕ is regular) and*
2) *the $N_x (x \in X)$ induced by N on the fibers of $X \underset{S}{\times} P^v$ over X are of degree 1. However [(or)]*[79] *to give an N invertible over the projective fibration $X \underset{S}{\times} P^v$ over X satisfying the condition 2) above is equivalent due to No. 1 to giving an invertible module L over X, N being determined as a function of L by $N = Lx_{O_S} O_{P^v}(1)$ and L being furthermore determined in terms of N by $L \approx pr_{1_*}(N(-1))$ where (-1) denotes the tensoring with $O_{P^v}(-1)$ over O_S.*

To give ϕ reduces to giving a section of $L \times O_P(1)$, i.e. a section of

$$pr_{1_*}(L \otimes_{\mathcal{O}_x} \mathcal{O}_{Xxp^v}(1)) = L \otimes_{\mathcal{O}_s} pr_{1_*}(\mathcal{O}_{Xxp^v}(1))$$

but because of III.2 we have $pr_{1_*}(\mathcal{O}_{Xxp^v}(1)) = E^v_X$ so that to give ϕ is equivalent to giving a morphism $g^*(E) \to L$ or, what is the same, a morphism $u\colon E \to g_*(L)$ (N.B.$g\colon X \to S$ is the canonical projection). It remains to explain the condition 1) above in terms of u. Since the constructions tht we made commute with base change it suffices to express this condition fiber by fiber and take into account that the points of P with value sin an extension k of $k(s)$ correspond exactly to straight lines in $E(s)\ x_{k(s)}k$ this condition can be expressed simply by requiring that for every $t \in E(s)$ the corresponding section $u(s)$ of L_s over X_s should be regular and that the analogous condition should be verified after every extension of the base field. We see as usual tht it suffices to test this condition over an algebraic closure of k. To summarize:

Proposition 4.1. *Let $g\colon X \to S$ be a flat morphism locally of finite presentation. Let $P = P(E)$ be a projective fibration over X defined by E locally free of finite type, everywhere*

[78] Ask A.G., is it EGA 21.3 [Tr]?
[79] [Fr]

$\neq 0$, *i.e., P has non-empty fibers, $P^v = P(E^v)$. Then there is a bijective correspondence between the set of linear systems of divisors over $X \mid S$ parametrized by P^v and the set of couples (up to isomorphism) (L, u) where L is an invertible module voer X and $u\colon E \to g_*(L)$ is a homomorphism such that for every $s \in S$ and for every point t of $E(s)x_{k(s)}k$ (for any extension k of $k(s)$ which we may suppose to be the algebraic closure of $k(s)$) the corresponding section $u(t)$ of $L_{s\,k}$ over $X_{s\,k}$ should be regular.*

We note that if the fibers of X are geometrically integral this condition on u means simply that for every $s \in S$, $u(s)\colon E(s) \to H^0(X_s, L_s)$ is injective a fact that we would also have to make explicit in 4.1 we would also have to recall (for convenience of reference) the construction of the divisor D in terms of (L, u) as the divisor of the evident section ϕ of $L \underset{O_S}{\mathrm{x}} O_{P^v}(1)$ defined by u.

Corollary 4.2. *Let us assume that $f\colon X \to S$ is proper flat and of finite presentation and with integral geometric fibers. Let L be an invertible module over X and $P = P(E)$ a projective fibration over S as in 5.1. There exists a module Q of finite presentation over S and an isomorphism of functors of the quasi-coherent O_S-module F: $\mathrm{Hom}(Q, F) \to g_*(L \underset{O_S}{\mathrm{x}} F)$. Once this is assumed,[80] the linear systems of divisors on X parametrized by P^v and associated to L in the sense of 4.1 correspond bijectively to* surjective *homomorphism $Q \to E^v$ modulo multiplication by a section of O_S^*. the existence of Q is reduced by a brief procedure to the case of S noetherian and in this case it is nothing else but III.7.7.6 of EGA III [Tr] (the hypothesis about the fibers of X being anyway useless). Since E is locally free of finit etype, to give a homomorphism $E \to f(L)$ is equivalent to giving a section of $L \underset{O_S}{\mathrm{x}} E^v$ therefore to giving a homomorhism of $Q \to E^v$. It remains to express that the condition mentioned in 4.1 is really erified, which (due to the hypothesis made about the fibers of X/S) is reduced to verifying that fiber by fiber the corresponding homomorphism*

$$E(s) \to H^0(X_s, L_s) \simeq \mathrm{Hom}_{k(s)}(Q(s), k(s))$$

is injective or again[81] $Q(s) \to E^v(s)$ is surjective which by Nakayama means also that $Q \to E^v$ is surjective. The "modulo sections of O_S" (or O_{S^*} P.B.) becomes "modulo isomorhisms" in 4.1. We may interpret 4.1 in another way by using the fact that $P(Q)$ represents the subfunctor of $\mathrm{Div}_{X/S}$ define dby L by virtue of No. 2. Consequently a linear system of divisors parametrized byP^v and associated to L is interpreted as a morphism $P^v = P(E) \to P(Q)$ the linear character of the family of divisors defined by such a

[80]Ceci pose [Fr]
[81]ou encore

morphism can be interpreted therefore by the fact that this morphism should be "linear", i.e. precisely defined by a surjective morphism of $Q \to E^v$. We see also in this case that the morhism $P^v \to \mathrm{Div}_{S/S}$ is a monomorphism (since $P^v \to P(Q) \to \mathrm{Div}_{S/S}$ is such) a fact anyway more general cf. corollary below. Let us therefore agree to say that two linear families of divisors of $X \mid S$ parametrized by the projective fibers P^v, $P^{v'}$ are isomorphic if they are transformed one into another by an S-isomorphism $P^v \to P^{v'}$ (which will be anyway unique due to the fact that we have a monomorhisms (?) in (into) $\mathrm{Div}_{S/S}$). We may therefore express 4.2 by saying that the set of classes up to isomorhism of linear systems of divisors over X associated to L is in bijective canonical correspondence with the set Grass(Q) (S) and this correspondence is compatible with any base change. We see that the functor: $S' \to$ set of classes (mod isomorphism) of linear systems of divisors of $X_{S'} \mid_{S'}$ associated to $L_{S'}$ is representable by the S-prescheme Grass(Q).

(Marginal Remarks Hard to Read, P.B.) [illegible ask AG]

We should make explicit in 4.1 tht $L \mid X - Z$ is canonically isomorphic to $f^*(O_P(1))$ (with the notations of the previus No.) so tht in this case $D \in \phi$ mentione din 3.3, $L \mid X - Z$ is nothing else but the canonical and unique extension of $f^*(O_P(1))$ to an invertible sheaf over X.

Proposition 4.3. *Let D be a linear system of divisors over $X \mid S$ parametrized by P^v where $g\colon X \to S$ is a flat morphism of finite presentation.*

- a) *Let us suppose that g is of finite presentation and that for every $x \in S$ if we denote by k an algebraic closure of $k(s)$ then there eixsts a prime cycle T associated to X_k such that $k \to H^0(T, O_T)$ should be an isomorhism (a conditioni automatically satisfied if g is proper and surjective). Then the morphism $D\colon P^v \to \mathrm{Div}_{S/S}$ is a monomorphism.*
- b) *Let us consider the map $u \to D \circ u$ of $Aut_S(P^v)$? into the set of families of divisors over X/S parametrized by P^v. Then if g is surjective the previous map is injective in particular $D = Du$ implies $u = id_{P^v}$ more generally the morphism of functors $Aut_S(P^v) \to$ Sys Lin $\mathrm{div}_{X/S}$, P^v is a monomorphism.*

We note that under the hypothesis of a), b) is a trivial consequence of a); on the other hand, b) is valid under less restrictive assumptions than a). We point out that a) becomes false if we abandon the restrictive hypothesis that we have made: take for example $S = \mathrm{Spec}\, k$, X an open subset of P^1_k not containing two distinct points a, b of $P^1_k(k)$. Then the two points a and b define the same divisors of X (savoir the zero divisor [Fr]) without being identical.

Let is assume first of all that S is the spectrum of a field k which we can evidently (by a "descent") assume to be algebraically closed. Let T be as in a) and we given it the induced reduced structure, we have then a morphism ("induced divisor")

$$\mathrm{Div}_{X/k} \to \mathrm{Div}_{T/k}$$

and it suffices to show that the composition

$$P^v \to \mathrm{Div}_{T/k}$$

is a monomorphism. Since the latter is again a linear system of divisors, we are reduced to the case $X = T$, thus to the case where $H^0(X, \mathcal{O}_X) \overset{\sim}{\longleftarrow} \mathcal{O}_S$. Thus for every S over k we have

$$g_{S*}(\mathcal{O}_{X_S}) \overset{\sim}{\longleftarrow}$$

thus if L over X and $u: E \to$??? are as in 4.1(???) if two sections ϕ andψ of E_S everywhere non-zero are such that $u(\phi)$ and $u(\psi)$ are sections of L_S over X_S having the same divisor then they are deduced from each other by multiplication by an invertible section of $\mathcal{O}_{X_S}$,it follows that ψ is deduced from ϕ by multiplication by an invertible section of $\mathcal{O}_S$ thus ϕ and ψ define the same point of P^v with values in S. Since every point of P^v with values in S is defined locally over S by a section ϕ of E_S which does not vanish (cf. Chap I) a) follows. To prove b) we note the:

Lemma.

Let D be a linear system of divisors over X non-empty and locally of finite type over k algebraically closed parametrized by $P^v(E)$ and let us consider the corresponding morphism

$$f: X - Z \longrightarrow P$$

where Z is the base locus (set of fixed points in the original [Tr]). Then if $r = \mathrm{rank}_k E > 0$ there exist $r+1$ points $x_i 1 \leq i \leq r+1$ of $X(k) - Z(k)$ such that the $f(x_i)$ give a "projective base" of P, i.e. such that for every subset $\mathcal{J}$ of $[1, r+1]$ having r elements the $f(x_i)$ are not containe din any hyperplane of P.

We may evidently suppose that $Z = \phi$. Since by 4.1 (ref ??? [Tr]) $f(X)$ is not contained in any hyperplane of P we conclude from the beginning the existence of r points $(1 \leq i \leq r)$ such that the $f(x_i)$ are projectively independent in P, i.e. are defined by linearly independent forms over E. It remains to prove that there exists an $x_{r+1} = x$ in $X(k)$ such that $f(x)$ is not in any of the r hyperplanes H_i defined by the system of $(r-1)$ from among the $f(x_i)$. But in the contrary case taking into account the "sorites" [Fr] of Jacobson we would have

$$f(X) \subset \bigcup_i H_i$$

thus if X_0 is an irreducible component of X then $f(X_0)$ would be contained in one of the H_i which contradicts 3.1 or 4.1 (ref??? [Tr]). This being established, to prove b) we may evidently suppose $Z = \phi$ and using 3.1 or 4.1 [Tr] we are reduced to proving that an automorphism u of P^v is determined if we know the composition of its contragradient u^v in P with $f\colon X \to P$ and that the analogous assertion is true after every base change $S \to \mathrm{Spec}(k)$ by an automorphism u of P^v_S. But this results immediately from the previous lemma and from the determination of automorphisms $P^v(E) = P(E^v)$ done in section (or number [Tr]) one, which implies that the effect of an automorphism of a projective fibration over an S is known (relative to a module (Module [Fr]) locally free of finite type) if we know its effect on a projective basis in each fiber.

Let us now summarize the general case: S arbitrary. Of course, after a base change over S we are reduced in a) to proving that any two sections of P^v over S which define the same divisor over X are identical and in b) to proving that any two automorphisms of P^v which are such that $D \circ u = D \circ v$ are identical. We may suppose that S is affine, the case b) where we do not suppose expressly that g is of finite presentation but g is surjective we reduce ourselves immediately (due to the fact that g is open) to the case where X is also affine thus of finite presentation over S. By a brief procedure we reduce to the case of S being noetherian.

Now for a noetherian base scheme S and for a morphism of functors $F \xrightarrow{h} G$ over S (F and G are the functors $(\mathrm{Sch}/S) \to (\mathrm{Ens})$) we have very general criteria which will be made explicit in Ch. V which allow to affirm that if for every $s \in S$ the corresponding morphism $F_s \to G_s$ is a monomorphism then $F \to G$ is a monomorphism (NB we put $F_s = Fx_S S_{k(s)}$ and the same for G_s), making simple assumptions about F and G (verified for example if F and G are both representable by preschemes of finite type over S, but in the case in hand only the first functor is representable à priori). We will summarize the argument of Ch. V in the two particular cases which are of interest to us here. We have two sections u, v (of P^v respectively of the projective group $GP(E^v)$) of a prescheme of finite type F over S about which we want to prove that they are equal. To do this it is clearly sufficient to prove that they are equal after the base change

$$\mathrm{Spec}(\mathcal{O}_S, s/\mathfrak{M}^{n+1}) \to S,$$

which reduces us to the case of S artinian and local. $S = \mathrm{Spec}\, A$. We proceed by induction on the integer n such that $\mathfrak{M}^{n+1} = 0$ which allows us to suppose tht the two sections u v are equal modulo $\mathfrak{M}^n$. Then one is induced from the other by means of an element δ of

$$\mathrm{Hom}_k(u_0^*(\Omega^1_{F_0/k}), V)$$

where $k = A\mathfrak{M}$ is the residual field, $F_0 = F \otimes_A k$ is the reduced fiber $V = \mathfrak{M}^n = \mathfrak{M}^n/\mathfrak{M}^{n+1}$. It suffices to prove that $\delta = 0$ using the hypothesis $h(u) = h(v)$.

The general principle of verification is as follows: to start with we express that $h(u)$ and $h(v)$ coincide modulo $\mathfrak{M}^n$ we see that their "difference" can be written as an element δ' of $\mathrm{Hom}_k(w_0^*(\Omega^1_{G_0/k}, V)$ where $w_0 = h_0(u_0) = h_0(v_0)$ and where $G_0 = G \times_A k$, this element is nothing else but the one deduced from δ by composition with the natural homomorphism

$$h_0^*\colon w_0^*(\Omega^1_{G_0/k}) \longrightarrow u_0^*(\Omega^1_{F_0/k})$$

deduced from $h_0\colon F_0 \to G_0$. Since $h(u) = h(v)$ thus $\delta' = 0$ the composition of δ with the preceding homomorphism h_0^* is zero so that we see that h_0 is surjective it follows that $\delta = 0$ and we are done. Now the fact that $h_0\colon F_0 \to G_0$ is a monomorphism thus inducing a monomorphism for the set of points with value sin the dual numbers over k implies that indeed h_0^* is surjective (its transpose being injective). This reasoning is valid since G is representable which is however not the case in the case that we consider. We can however define a vector bundle $\mathcal{G}_{w_0}$ over k playing the role dual to $w_0^*(\Omega^1_{G_0/k})$ (illegible) tangent to G_0 at w_0 by expressing the "deviation" of two points of G which coincide modulo $\mathfrak{M}^n$ as an element of $\mathcal{G}_{w_0} \otimes_k V$. This is essentially straightforward and is contained in the systematic developmentsof par. 26 (? Infinitesimal extensions) which we review here. In the case a) G is the functor $\mathrm{Div}_{X/S}$ w_0 corresponds to a Cartier divisor D_0 over $X_0 = X \otimes_A k$ and we have to take $\Omega = H^0(D_0, n_{D_0/X_0})$ where $n$is the normal sheaf to D_0 in X_0, isomorphic also to the induced sheaf on D_0 by $\mathcal{O}_{X_0}(D_0)$ on D_0. In the case b) we may suppose that D has no fixed points and it is more convenient to interpret the situation in terms of morphisms into P (see 4.1) so that G becomes the functor

$$\mathrm{Hom}_S(X, P)$$

and G_{w_0} should be the space

$$\mathrm{Hom}_{\mathcal{O}_{X_0}}(f_0^*(\Omega^1_{P_0/k}, \mathcal{O}_{P_0})$$

In both cases we have a natural homomorphism

$$G_{u_0} \otimes_k V \to G_{w_0} \otimes_k V$$

(where G_{u_0} is the dual fo $u_0^*(\Omega^1_{F_0/k}))$ expressing the passage from the deviation δ to the corresponding deviation δ' by the mapping h and the injectivity of this mapping results from the injectivity of $G_{u_0} \to G_{w_0}$ which 'elle provient du fair [Fr] that $h_0\colon F_0 \to G_0$ is a monomorphism.

Practially it does not seem possible to write up that last part of the proof without referring to the small calculations of paragraph 25 (which it is out of the question to redo here in the particular case). We note that this does not give rise to a vicious circle since the par. 25 and the calculations that we have developed only depend on the rewrite of differential calculus from par. 16 and also 4.3 will not be used gain in Ch. IV except perhaps in the two following numbers or sections [Editors Note: Did Grothendieck intend this part as fragment of EGA IV, this seems very likely].

The interest of 4.3 a) is to prove that under the stated conditions the parametrizing projective fibration can be interpreted intrinsically the notion of class (up to an isomoprhism over the projective fibration parametrizing ???) of the linear system over X/S as being a subfunctor P^v of $\mathrm{Div}_{X/S}$ which satisfies certain properties (savoir [Fr] is representable by a projective fibration and the family of divisors defined by the canonical injection of the latter into $\mathrm{Div}_{S/S}$ is linear in the sense of No. 3), which is essentially the classical point of view (where a linear system of divisors is defined as a *set* of divisor ssatisfying certain conditions, compare 4.4). On the other hand 4.3 b) is equivalent to saying that if g is surjective then if two linear system sof divisors over X/S parametrized by two projective fibrations $P^v(P^v)'$ are isomorphic then there exists a unique isomoprhismf rom P^v to $(P^v)'$ (compatible with D and D') we may therefore say that a class (up to isomoprhism) of linear systems over S/X determines it parametrizing projective fibration *up to a unique isomoprhism.* Technically this result will allow us (once we have the descent theory of Chapter V [Editor - not yet written also numbering is of only historical interest]) to make *the faithfully flat descent for linear systems of divisors* – under the reservation always however to allow also as parametrizing fibrations “the twisted projective fibrations” which will be done in a future section.

Descending again to the earth, and even lower, to explain in vulgar terms the notion of a linear system we place ourselves for simplicity over a box field (although the statement will hold essentially as such over an affine base)

Proposition 4.4. *Let X be a prescheme of finite type over a field k, such that*

$$k \longrightarrow H^0(X, \mathcal{O}_X)$$

is an isomoprhism. To every linear system D of divisors over X parametrized by a projective fibrtion P^v over k we associate the set (!) Ens(D) of all the divisors over X of the form $D(t)$ where $t \in P^v(k)$

- a) *If d' is another linear system of divisors over X parametrized by a projective fibration $(P^v)'$ then D and D' are isomorphic if and only if Ens$(D) =$ Ens(D').*

b) *Suppose k alg. closed or X geometrically integral.*

In order that the set Δ of positive Cartier divisors over X should be of the form Ens(D) it is necessary and sufficient that there should exist a k-subspace of the vector space E of meromorphic functions on X such that for every $\phi \in E - (0)$ ϕ shuld be regular, i.e. div(ϕ) is defined and that Δ should be the set of div(ϕ) for $\phi \in E - (0)$

c) *Let E, E' be two k-vector subspaces of the meromorphic functions on X satisfying the assumption of b) then the sets of divisors Δ, Δ' defined by them are equal iff there exists a regular pseudo-function ϕ over X such that*

$$E' = \phi E.$$

If $E \neq (0)$, i.e. $\Delta \neq \phi$ such a ϕ is defined modulo multiplication by an element of k^8.

The proof is an easy exercise using 4.1 and I dispense with writing down the proof except if you protest this. In addition, it seems to me that 4.4 could profitably come before 4.3, being technically more trivial. Note also that if X is geometrically integral the condition on E stated in b) becomes *void*. The restriction made at the beginning of b) is attached to the fact that otherwise the condition announced for b) may not be true after passing to the alg. closure of k (it is easy to give examples) in every characteristic even if k is separably closed in char $p > 0$. For good measure we would have to announce b) without supplementary conditions on k or X but by announcing the condition over E and by passing to the algebraic closure of k (and noting that if X is geometrically integral this condition becomes void). By abuse of language, a *set* Δ of divisors of the form Ens(D) will be often called a linear system of divisors on X (slightly illegible)
[Editor: Here the original notes of Grothendeick end]

Finis opus coronat [Ed.]

Part IV

Section 1

Letter to Diéudonne 29/9/1965

Dear Diéudonne,

Thank you for the letter of the 24th and for the table of contents of par. 16 to 19. I would be happy to receive one day the tentative table of contents for par 20 and 21. It's ok to adjoin them to volume 4 of Ch. IV. But how are you going to subdivide my old par. 20 and what will be the titles of the two parts?

Since I am beginning to be lost in the plan and it is often convenient to be able to refer to (without saying too many stupid things)[82] to a number in a paragraph, I give you here what seems to me to be the actual plan, tell me if you agree.

20 ???
21 ???
22 Linear systems complements about the Picard group
23 GRASSMANIANS
24 Smooth forms ordinary quadratic singularities
25 Hyperplane sections et bordel[83] [Fr]
26 Resultant and discriminant
27 Infinitesimal extensions

The 25th is at risk in addition of being too long and you may wish to subdivide it into two. Still $27 = 3^3$ is a very pretty number!

It is out of the question that I should publish the appendix to para. 18 under my name. Your formulation (writeup) has almost nothing in common with the vague manuscript notes that I sent to you.

and limiting myself to saying: Even if I had given you any you just have to do the same as for que for complete rings...

It would be on the other hand a pity if your work about its formal setting should be lost for the possible users (il finit toujours par s'en trouver...) There can always be some to be found.

That is why I ask you to reconsider the question of making a 'joint paper.'[84]

As for par. 20, 10.9.1 it is of course necessary to use the fact that the set of points of Z_λ where F_λ restricted to the fiber is of the depth $> n$ given is constructible (we have to prove the same meme in par. 12 that it is open with the assumption of flatness and of finite presentation which we make). Since its inverse image in Z is everything [Fr] that is already a little further than than λ. This is really always the same argument qui revient!

That repeats itself.

Bien a toi (all the best) A. Grothendieck

[82](original is in off-color French)
[83]and the rut of the shiff(   )(original is in off-color French)
[84]crossed out in the original from this point till bien a toi [Tr]