8 May 2007

Dear Barry, William, and Bobby,

Recently, I tried to address some of the questions Barry brought up in his last email. In particular:

- Does the dependence of $e_a^b(E)$ on the elliptic curve E involve the rank r of the elliptic curve E? If so, we might denote the exponent $e_a^b(r)$.
- Is $e_a^b(E)$ independent of a and b? If so, we might denote the exponent $e(E)$.

Provided are some plots of data suggesting the following:

- $e_a^b(E)$ varies extremely little across elliptic curves of the same rank.
- $e_a^b(E)$ varies somewhat across different intervals suggesting that certain sets of the a_p's converge slightly faster / slower than other sets. Exactly how so is still unclear.

1 Does $e_a^b(E)$ vary across E with fixed rank?

Each of the Figures 1-4 below sample nine elliptic curves. The nine curves in each figure share a fixed rank. Each plot is over the interval $(-1, 1)$ with cutoff $C = 10^6$. Some observations and notes:

- The first curve in each figure is that with the smallest conductor of the given rank.
- Figure 1 features $e_a^b(E)$ for three curves of rank 0 with complex multiplication. I left them there to show that convergence is fairly uniform even in this case.
- $e_a^b(E)$ fluctuates only slightly across each chosen rank. Given that I can now look up curves of even larger rank, thanks to William's implementation of a larger elliptic curve database in SAGE, more plots can be made readily available for various curves of higher rank.
- Perhaps it may be useful to look at classes of elliptic curves with equal torsion subgroups, as suggested in Barry's previous email. However, this data taken over a wide variety of curves, suggests that there may not be much of a difference.

Figure 1: $e_a^b(E)$ for nine different elliptic curves of rank 0 over $(-1, 1)$, $C = 10^6$.

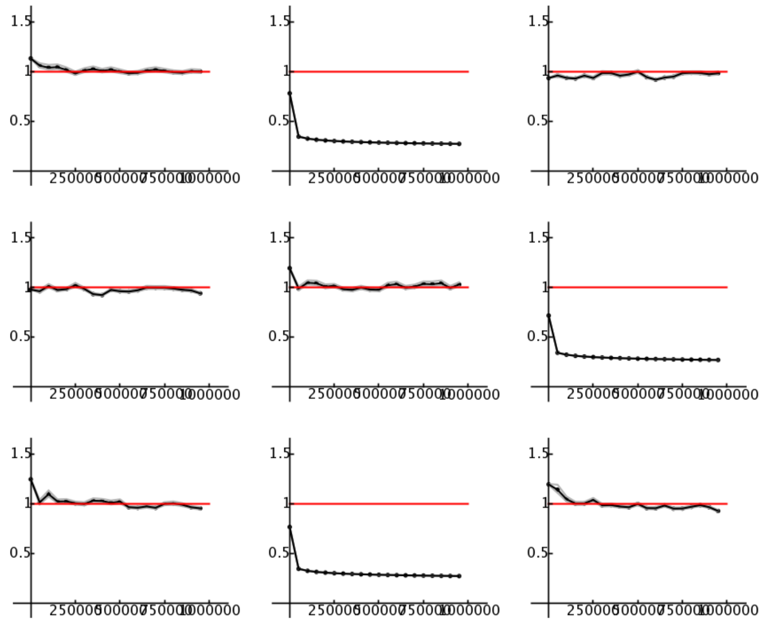


Figure 2: $e_a^b(E)$ for nine different elliptic curves of rank 1 over $(-1, 1)$, $C = 10^6$.

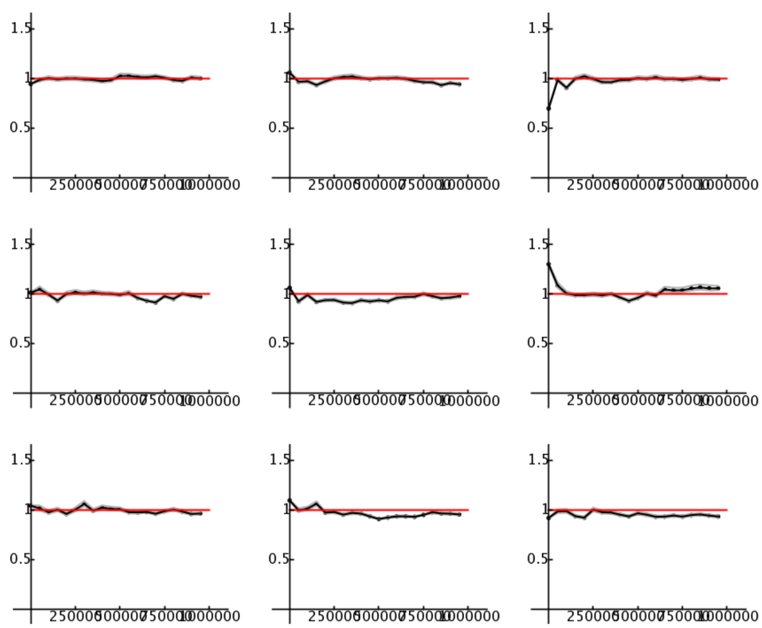


Figure 3: $e_a^b(E)$ for nine different elliptic curves of rank 2 over $(-1, 1)$, $C = 10^6$.

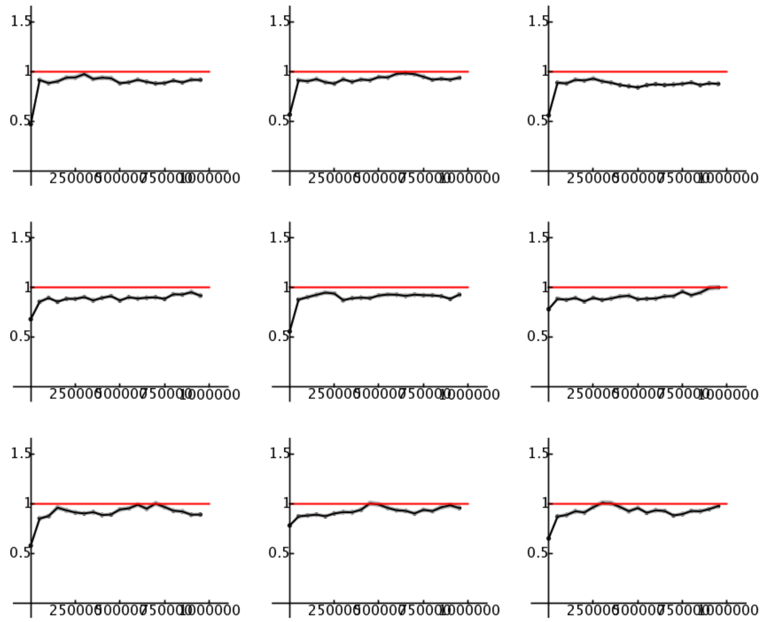
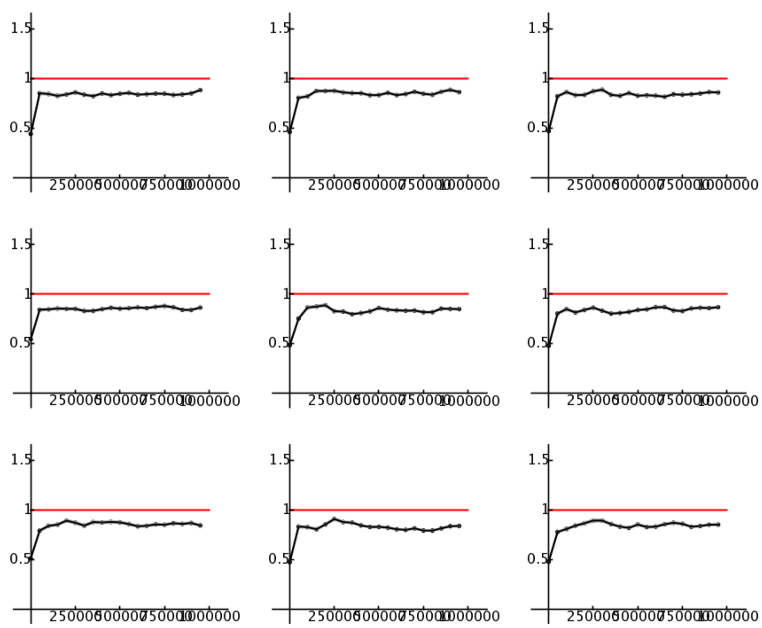


Figure 4: $e_a^b(E)$ for nine different elliptic curves of rank 3 over $(-1, 1)$, $C = 10^6$.



2 Does $e_a^b(E)$ vary across (a, b)?

Given the data provided below, it's a bit more difficult to conjecture anything about the convergence of $e_a^b(E)$ within different intervals. I first partitioned the interval $(-1, 1)$ into four equal parts: $(-1, -1/2), (-1/2, 0), (0, 1/2)$, and $(1/2, 1)$. For eight elliptic curves of ranks $0 - 8$, each with smallest conductor of the corresponding rank, I plotted $e_a^b(E)$ over each of the subintervals above with cutoff $C = 3 \cdot 10^6$. The following figures are laid out like this:

Plot of $e_a^b(E)$ for E of rank n with smallest conductor:

over $(-1, -\frac{1}{2})$	over $(-\frac{1}{2}, 0)$
over $(0, \frac{1}{2})$	over $(\frac{1}{2}, 1)$.

Some observations and notes:

- In some cases, the value of $e_a^b(E)$ is the same across all intervals at $C = 3 \cdot 10^6$. However, the differences in fluctuation along the way is, to me, a bit peculiar. I'm not sure what to make of the data.
- What does it mean for $e_a^b(E)$ to be larger / smaller across one interval than another? How can the a_p's converge faster / slower in one interval instead of another?
- The a_p histograms show show that there is a skew to the left for elliptic curves of high rank and low cutoff C. Is this reflected in the following plots? How can this behavior be reflected in the first place?

3 Where to go from here?

I'd like to pursue more interval-related questions. Again, is there some way to quantify the a_p skew as shown in the histograms? Also, I'll try varying the interval sizes to see if any interesting changes occur in the data. It was suggested a while ago that the arrangement of zeros on the critical line of the corresponding L function for each elliptic curve may share some sort of link with $e_a^b(E)$. One of my goals for the next week or so is to ponder this notion. Finally, seeing that we could probably write $e_a^b(E)$ as $e(r)$ I'll conjure up a "step-graph" for $r = 0, \ldots, 8$.

Peace,

Chris

Figure 5: $e_a^b(E)$ for E with rank 0 over partitioned interval, $C = 3 \cdot 10^6$.

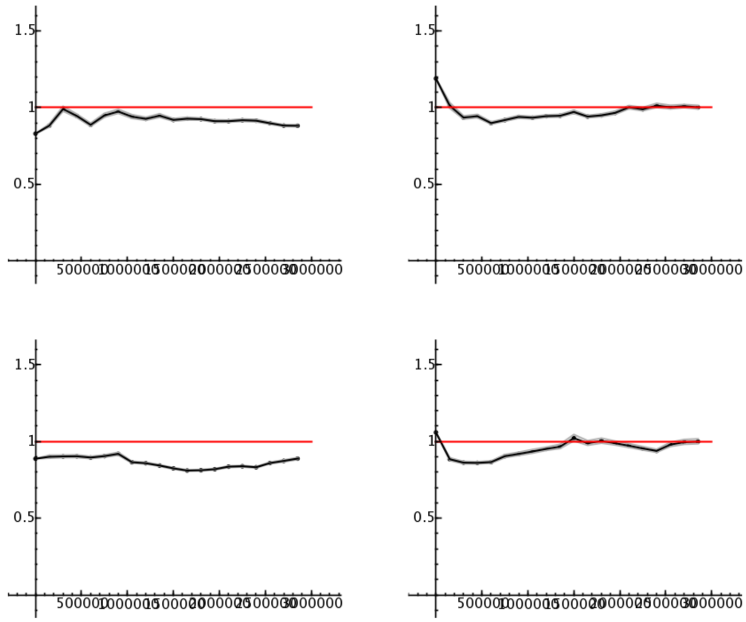


Figure 6: $e_a^b(E)$ for E with rank 1 over partitioned interval, $C = 3 \cdot 10^6$.

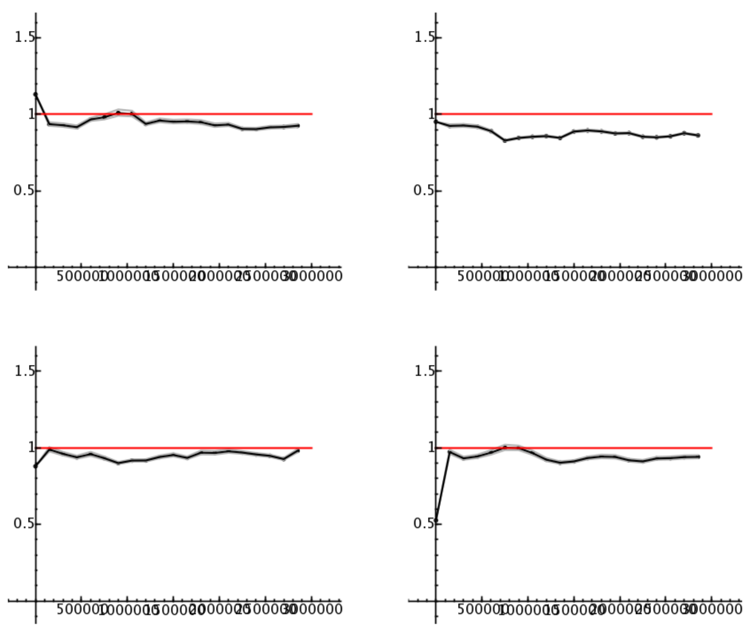


Figure 7: $e_a^b(E)$ for E with rank 2 over partitioned interval, $C = 3 \cdot 10^6$.

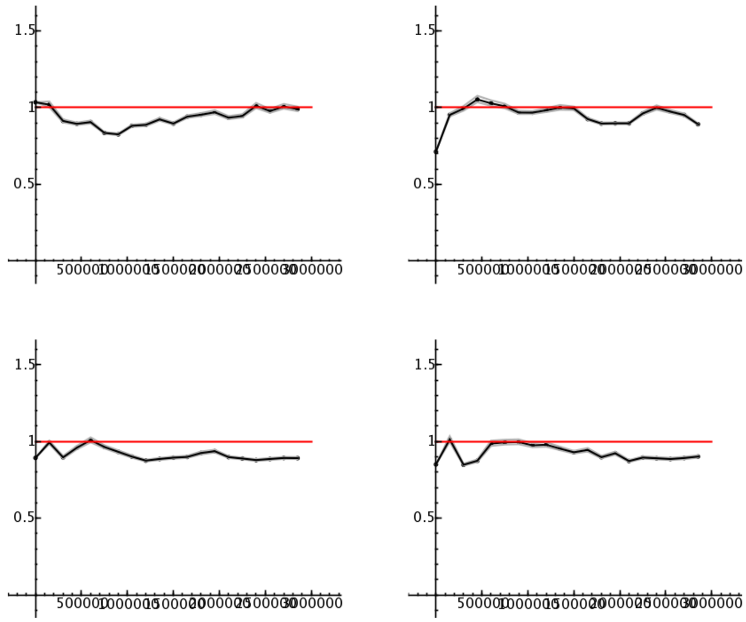


Figure 8: $e_a^b(E)$ for E with rank 3 over partitioned interval, $C = 3 \cdot 10^6$.

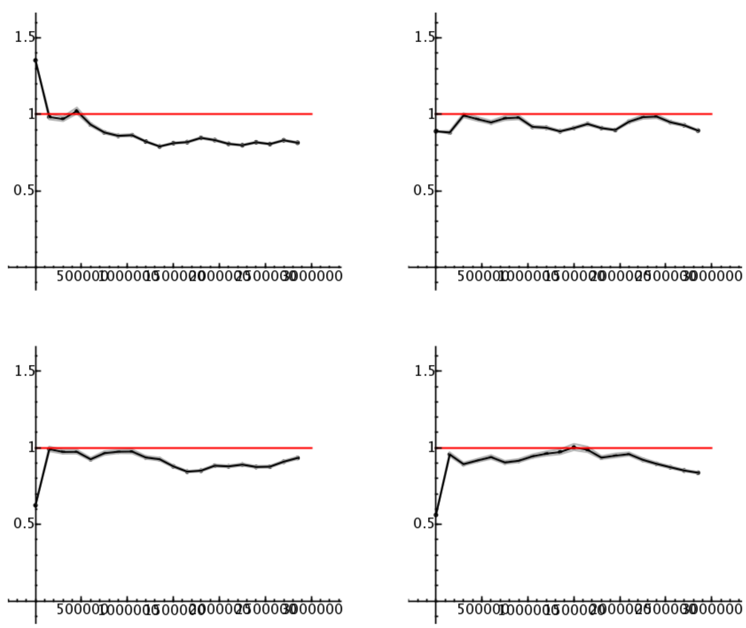


Figure 9: $e_a^b(E)$ for E with rank 4 over partitioned interval, $C = 3 \cdot 10^6$.

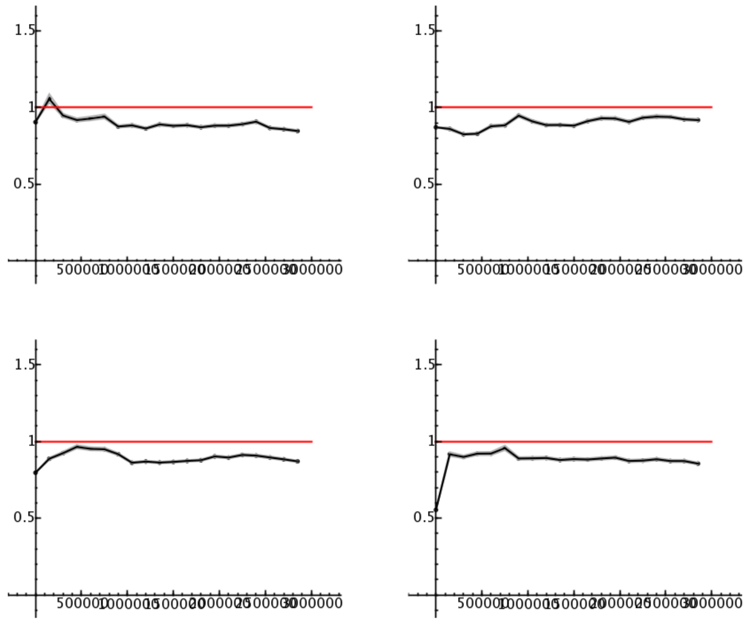


Figure 10: $e_a^b(E)$ for E with rank 5 over partitioned interval, $C = 3 \cdot 10^6$.

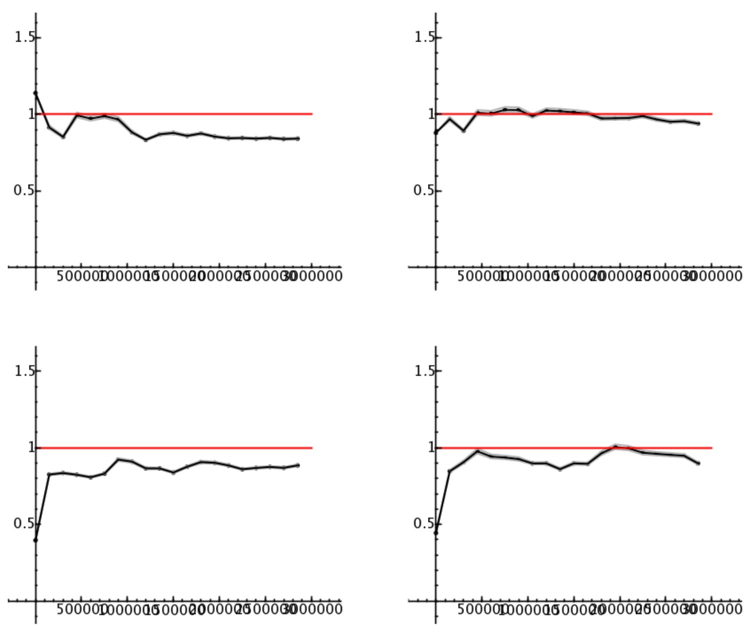


Figure 11: $e_a^b(E)$ for E with rank 6 over partitioned interval, $C = 3 \cdot 10^6$.

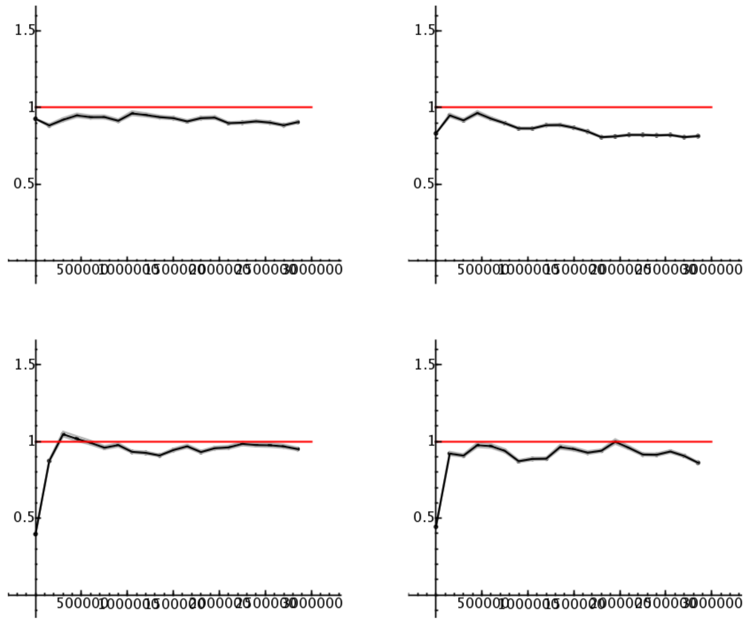


Figure 12: $e_a^b(E)$ for E with rank 7 over partitioned interval, $C = 3 \cdot 10^6$.

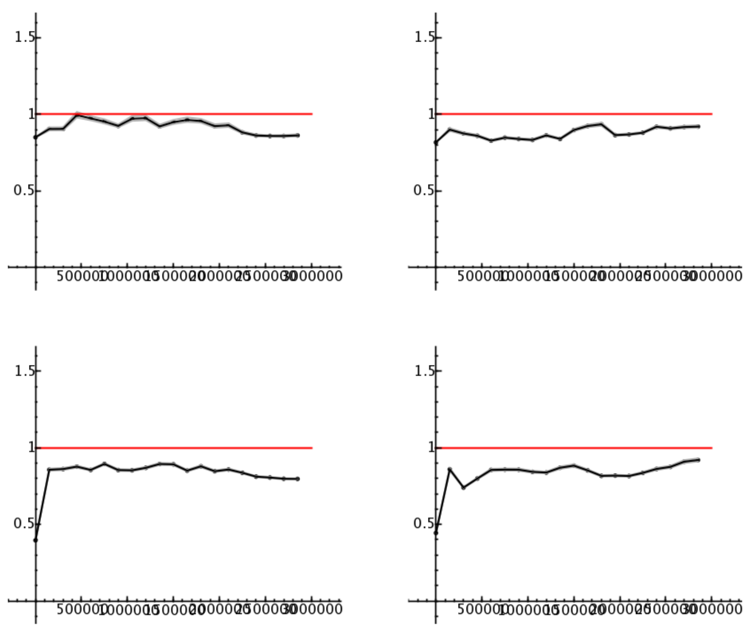


Figure 13: $e_a^b(E)$ for E with rank 8 over partitioned interval, $C = 3 \cdot 10^6$.