Let $r_o, \ldots, r_p$ be coset representatives for $\Gamma_0(p)$ in $\mathsf{SL}_2(\mathbf{Z})$. So

$$\mathsf{SL}_2(\mathbf{Z}) = r_o\Gamma_0(p) \cup r_1\Gamma_0(p) \cup \cdots \cup r_p\Gamma_0(p).$$

E.g.,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let V be the $p+1$-dimensional vector space with basis

$$x_0, x_1, \ldots, x_p.$$

Theorem (Manin). There is an isomorphism

$$V/R = \mathrm{ModSym}(\Gamma_0(p)),$$

where x_i maps to $r_i\{0,\infty\} = \{r_i(0), r_i(\infty)\}$, and the relations R are described explicitely below.

The subspace R of relations is the subspace generated by all

$$x_i + Sx_i$$

$$x_i + Tx_i + T^2 x_i,$$

where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Generating modular symbols:

$$\{0,\infty\},\ \{0,1\},\ \{0,1/2\},\ldots,\{0,1/10\},\ \{\infty,0\}$$

Doing the linear algebra, we find that $\{0,\infty\}$, $\{0,1/5\}$, $\{0,1/7\}$ are a basis. And, e.g.,

$$\begin{aligned}
\{0,1/2\} &= -\{0,1/5\}\\
\{0,1/3\} &= -\{0,1/7\}\\
\{0,1/4\} &= \quad \{0,1/5\}-\{0,1/7\}
\end{aligned}$$