Homework 1 for Math 581F, Due October 10, 2007

The above date is not a typo. Each problem has equal weight, and parts of problems are worth the same amount as each other.

1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.
   (a) Find the Smith normal form of A.
   (b) Prove that the cokernel of the map $\mathbb{Z}^3 \to \mathbb{Z}^3$ given by multiplication by A is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}$.
2. Show that the minimal polynomial of an algebraic number $\alpha \in \overline{\mathbb{Q}}$ is unique.
3. Which of the following rings have infinitely many prime ideals?
   (a) The integers $\mathbb{Z}$.
   (b) The ring $\mathbb{Z}[x]$ of polynomials over $\mathbb{Z}$.
   (c) The quotient ring $\mathbb{C}[x]/(x^{2005} - 1)$.
   (d) The ring $(\mathbb{Z}/6\mathbb{Z})[x]$ of polynomials over the ring $\mathbb{Z}/6\mathbb{Z}$.
   (e) The quotient ring $\mathbb{Z}/n\mathbb{Z}$, for a fixed positive integer n.
   (f) The rational numbers $\mathbb{Q}$.
   (g) The polynomial ring $\mathbb{Q}[x, y, z]$ in three variables.
4. Which of the following numbers are algebraic integers?
   (a) The number $(1 + \sqrt{5})/2$.
   (b) The number $(2 + \sqrt{5})/2$.
   (c) The value of the infinite sum $\sum_{n=1}^{\infty} 1/n^2$.
   (d) The number $\alpha/3$, where α is a root of $x^4 + 54x + 243$.
5. Prove that $\overline{\mathbb{Z}}$ is not noetherian.